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ESSENTIALS OF

ANALYTIC GEOMETRY

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PREFACE

Analytic Geometry, as a college course in mathematics, has been materially changed during the last twenty-five years. Many of the things that were formerly included have been carried forward into the preceding subjects, college algebra and trigonometry. Students now enter a formal course in analytic geometry with a fair understanding of rectangular co-ordinates and of the graphing of elementary algebraic and transcendental functions.

The present volume is the outcome of a serious attempt to meet the current demand for a brief text that is adapted to the changed conditions and at the same time is sufficiently complete, adequate for the curricular needs of the student, and above all a teachable text. In pursuing our aims we have tried to make use of the information the student already possesses and have felt compelled to omit some topics, interesting in themselves, and yet of only secondary importance in so far as future needs are concerned.

In a text for a course as well standardized as analytic geometry, it would be too much to expect any startlingly new material. Perhaps the nearest approach to this would be the chapter on empirical equations, which was included with the idea of making the subject matter of more value to future scientists who otherwise would generally have no opportunity to become acquainted with this important subject in the ordinary undergraduate curriculum.

One point of interest is the more than average use of graphic methods as a labor-saving device.

The exercises of the text afford ample opportunity for drill, while reviews are inserted in places where it was felt they would be most effective. As a general policy answers are given for the odd-numbered exercises and to some others that may offer particular difficulty or interest.

The summaries of formulas are presented in the hope that they will make the book more useful for reference purposes.

For the benefit of those instructors who may prefer a slightly altered sequence of topics, Chapter IX, on general methods of curve tracing, was written independently of the discussion of conics so that it may precede Chapter III without confusion.

We hope we have succeeded in producing a book that is readable and intellectually stimulating, one that will make a real contribution toward more effective teaching of and a greater interest in mathematics.

We wish to express our thanks to Professor E. R. Smith for his kindly criticism and helpful suggestions for the improvement of the work, to Professor W. L. Porter of The Agricultural and Mechanical College of Texas, whose criticism has aided us in strengthening the treatment, and to The Dryden Press, whose co-operation makes possible the appearance of the book in its present form. We also acknowledge with thanks the suggestions of Professor H. A. Fisher of the North Carolina State College, Professor J. H. Bushey of Hunter College, New York, Professor H. E. Stelson of Kent State University, and Professor Daniel E. Whitford of the Polytechnic Institute of Brooklyn, who read the proofs of this work.

Harvey P. Pettit
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CHAPTER

I

FORMULAS FOR REFERENCE

In the study of analytic geometry, there will be frequent occasions when it will be necessary to recall certain theorems, formulas, and processes developed in earlier courses in mathematics. We give in the next few pages a brief summary of the most essential of this information.

ALGEBRA

1 The roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The roots are real and unequal if $b^2 - 4ac > 0$.
The roots are real and equal if $b^2 - 4ac = 0$.
The roots are imaginary if $b^2 - 4ac < 0$.

2 Simultaneous solution of two equations of the first degree

$$Ax + By + C = 0,$$
$$Dx + Ey + F = 0.$$

$$x = \frac{\begin{vmatrix} -C & B \\ -F & E \end{vmatrix}}{\begin{vmatrix} A & B \\ D & E \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} A & -C \\ D & -F \end{vmatrix}}{\begin{vmatrix} A & B \\ D & E \end{vmatrix}}, \quad \text{or}$$

$$x : y : 1 = \begin{vmatrix} B & C \\ E & F \end{vmatrix} : -\begin{vmatrix} A & C \\ D & F \end{vmatrix} : \begin{vmatrix} A & B \\ D & E \end{vmatrix},$$

where $\begin{vmatrix} A & B \\ D & E \end{vmatrix}$ is understood to mean $AE - BD$ and not zero.

E.g., $3x - 2y + 3 = 0$, $2x + 4y - 14 = 0$, give

$$x : y : 1 = \begin{vmatrix} -2 & 3 \\ 4 & -14 \end{vmatrix} : -\begin{vmatrix} 3 & 3 \\ 2 & -14 \end{vmatrix} : \begin{vmatrix} 3 & -2 \\ 2 & 4 \end{vmatrix} = 16 : 48 : 16,$$

or $x : y : 1 = 1 : 3 : 1$. Hence $x = 1$, $y = 3$.

3 Determinant of the third order

$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & K \end{vmatrix}$ is understood to mean

$$AEK - AFH - BDK + BFG + CDH - CEG,$$

which can be expressed readily as

$$A \begin{vmatrix} E & F \\ H & K \end{vmatrix} - B \begin{vmatrix} D & F \\ G & K \end{vmatrix} + C \begin{vmatrix} D & E \\ G & H \end{vmatrix}.$$

Notice that each second-order determinant in this expansion is found by dropping the row and column of the element by which it is multiplied, and that these elements are taken, in turn, from the same row or column and are preceded alternately by the plus and minus signs.

Thus,

$$\begin{vmatrix} 3 & 5 & 9 \\ 2 & 1 & 7 \\ 1 & 2 & 8 \end{vmatrix} = 3 \begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix} - 5 \begin{vmatrix} 2 & 7 \\ 1 & 8 \end{vmatrix} + 9 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 3(8 - 14) - 5(16 - 7) + 9(4 - 1)$$

$$= -18 - 45 + 27 = -36.$$

Again

$$\begin{vmatrix} 5 & -3 & -2 \\ 1 & 2 & 1 \\ -2 & 1 & 4 \end{vmatrix} = 5 \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix} + (-2) \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}$$

$$= 5(8 - 1) + 3(4 + 2) - 2(1 + 4)$$

$$= 35 + 18 - 10 = 43.$$

GEOMETRY

1 The area of a triangle is one-half the product of the base and altitude.

2 The area of a trapezoid is one-half the product of the sum of the bases and the altitude.

3 Pythagorean theorem: The square on the hypotenuse of a right triangle is equal to the sum of the squares on the legs.

4 The locus of points equidistant from two fixed points is the perpendicular bisector of the line segment joining them.

5 The locus of points equidistant from two intersecting lines is the bisectors of the angles between the lines.

6 The locus of points in a plane and at a fixed distance from a given point of the plane is a circle with the point as center and the fixed distance as radius.

7 The volume of a prism is the product of the altitude and a right cross section.

8 The volume of a truncated prism is the area of a right cross section multiplied by the sum of the parallel edges and divided by the number of parallel edges.

9 The volume of a pyramid is one-third the product of the altitude and the area of the base.

TRIGONOMETRY

1 Values of functions of certain convenient angles

Angle	0	$\pi/6 = 30°$	$\pi/4 = 45°$	$\pi/3 = 60°$	$\pi/2 = 90°$	$\pi = 180°$	$3\pi/2 = 270°$
Sine	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	0	-1
Cosine	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	-1	0
Tangent	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞

2 Fundamental identities

$$\csc\theta = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \cot\theta = \frac{1}{\tan\theta},\ \tan\theta = \frac{\sin\theta}{\cos\theta},$$

$$\sin^2\theta + \cos^2\theta = 1,$$
$$\sec^2\theta = 1 + \tan^2\theta,$$
$$\csc^2\theta = 1 + \cot^2\theta.$$

3 Other formulas

$$\sin(\pi/2-\theta) = \cos\theta,\ \cos(\pi/2-\theta) = \sin\theta,\ \tan(\pi/2-\theta) = \cot\theta,$$
$$\sin(\pi-\theta) = \sin\theta,\ \cos(\pi-\theta) = -\cos\theta,\ \tan(\pi-\theta) = -\tan\theta,$$
$$\sin(\pi+\theta) = -\sin\theta,\ \cos(\pi+\theta) = -\cos\theta,\ \tan(\pi+\theta) = \tan\theta,$$
$$\sin(-\theta) = -\sin\theta,\quad \cos(-\theta) = \cos\theta,\quad \tan(-\theta) = -\tan\theta,$$
$$\cos(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2,$$
$$\tan(\theta_1-\theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1\tan\theta_2},$$
$$\sin 2\theta = 2\sin\theta\cos\theta,$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta,$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta},$$
$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}},\ \cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}},$$
$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

4 Area of triangle $ABC = \frac{1}{2}ab\sin C$.

5 Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cos A$.

CHAPTER

II

POINTS

II.1 CO-ORDINATE SYSTEMS

The study of analytic geometry is the application of algebra to geometry, in which we attack the twofold problem of finding an algebraic representation of geometric properties of figures and of finding the geometric interpretation of certain algebraic forms and operations. The first problem is to set up a correspondence between sets of numbers and points, in other words, to establish systems of co-ordinates. There are many such systems of co-ordinates, but we shall limit our consideration to the two most commonly used in elementary mathematics, **rectangular co-ordinates** and **polar co-ordinates.**

II.2 RECTANGULAR CO-ORDINATES

In our study of algebra and trigonometry, we have become familiar with the rectangular co-ordinate system, in which the

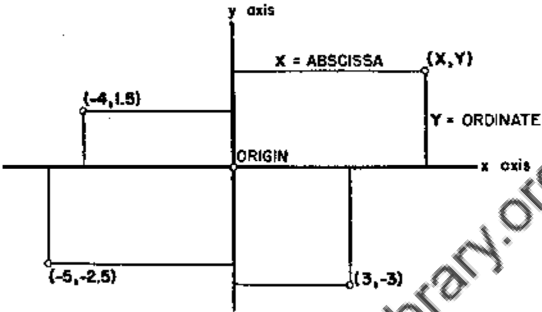

FIG. 1.—RECTANGULAR CO-ORDINATES.

x- and y-axes are at right angles to each other (Fig. 1). Corresponding to each point in the plane there is a pair of numbers (x,y), called respectively the **abscissa** and the **ordinate** of the point. These are the directed distances from the x- and y-axes, respectively. The x-measurements start from the y-axis, positive to the right, negative to the left. The y-measurements start from the x-axis, positive up, negative down. The intersection of the axes is called the **origin.** The axes divide the plane into four **quadrants,** numbered as shown in Fig. 2.

The ordinate of every point of the x-axis is zero, the abscissa of every point on the y-axis is zero, and the co-ordinates of the

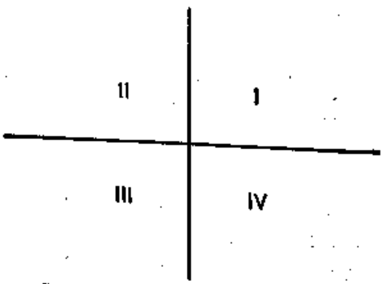

FIG. 2.—THE QUADRANTS.

origin are both zero. Each pair of values, (x,y), corresponds to a single point, and to every point there corresponds a unique pair of values, (x,y).

EXERCISES

1. What are the algebraic signs of x and y in the various quadrants?
2. Locate the following points:
   $(2,4)$, $(3,-2)$, $(-5,3)$, $(-3,-2)$, $(-\frac{1}{2},3)$, $(1.6,-2.8)$, $(4,-\frac{2}{3})$, $(-5\frac{1}{4},0)$.
3. Read the co-ordinates of the indicated points of Fig. 3 with respect to the given axes.

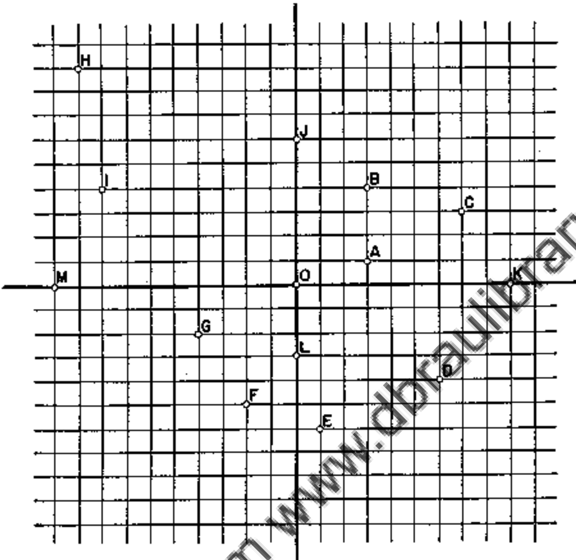

FIG. 3.

4. Show how the rectangular co-ordinate system could be applied to house numbering in a city. Is the system in use in any city with which you are familiar?
5. How is a point on the surface of the earth located? Does the system of reference lines resemble the rectangular co-ordinate system? In what way does it differ?
6. How would you describe the location of the capital city of your state, with reference to the state map?

II.3 POLAR CO-ORDINATES

In the polar co-ordinate system, points are located by giving the distance and direction from a fixed point, called the **pole.**

In order to determine the direction of the point from the pole we need a fixed line of direction, known as the **polar axis.** We take a line segment whose length is equal to the distance of the point from the pole, and imagine this segment, with one end fixed at the pole, turning through a definite angle from the polar axis, counterclockwise for a positive angle, clockwise for a negative angle, until the other end coincides with the given point. The angle generated in this manner is called the **vectorial angle,** and the line segment from the pole to the given point is the **radius vector.**

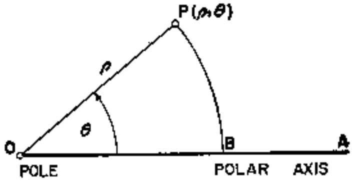

FIG. 4.—POLAR CO-ORDINATES.

In Fig. 4, O is the pole, OA the polar axis, OP the radius vector which we think of as having turned through the positive angle θ from the position OB. The direction OA is considered positive on the polar axis. If, for the same value of θ, the radius vector were negative, the point would lie on the line OP extended through and beyond O, as shown in Fig. 5.

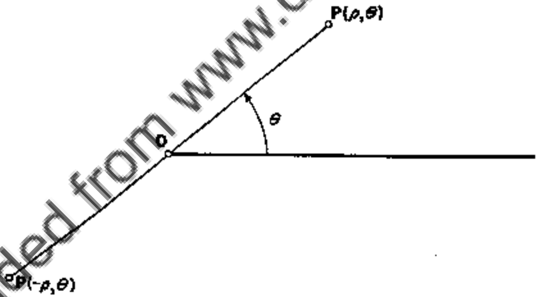

FIG. 5.—POLAR CO-ORDINATES.

Just as in rectangular co-ordinates, corresponding to each pair of co-ordinates (ρ,θ), there is a single point. The converse statement is not true, since the addition or subtraction of any multiple of 360° gives a new vectorial angle for the same point. Thus (ρ,θ), $(\rho,\theta + 360°)$, $(\rho,\theta - 720°)$ all correspond to the same point. Similarly, changing the vectorial angle by 180° and, at the same time changing the sign of the radius vector gives new polar co-ordinates for the same point. Thus (ρ,θ), $(-\rho,\theta + 180°)$, $(-\rho,\theta - 180°)$ represent the same point.

EXAMPLE. Locate the points (2,30°), (−3,45°), (3,225°), (2,−90°), in Fig. 6.

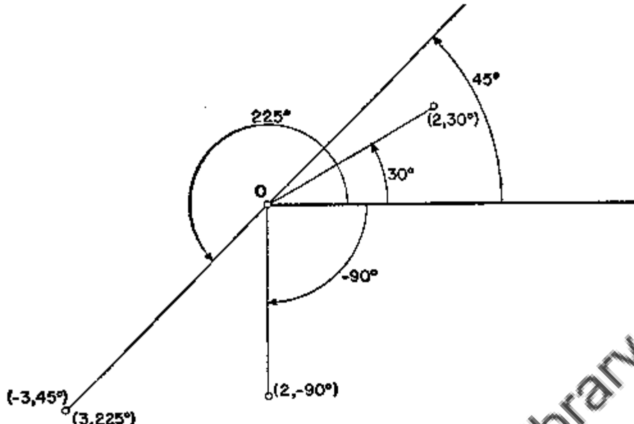

FIG. 6.—POLAR CO-ORDINATES.

EXERCISES

1. Locate the following points:
   (a) (1,15°), (−2,−150°), (−6,π/6), (5,−π/3), (1,−π/2). (Remember that a radian is the central angle whose intercepted arc is equal to the radius. Thus 1 radian = 57.29° [approx.], and 180° = π radians.)
   (b) (4,−⅔), (−5,0), (9,π), (−3.5,π/4), (−4,225°).
   (c) (−6,0), (1,−120°), (−6,120°), (7,3π/4), (1,π).
2. Read the polar co-ordinates of the indicated points in Fig. 7. The angular divisions marked are at 15° intervals.

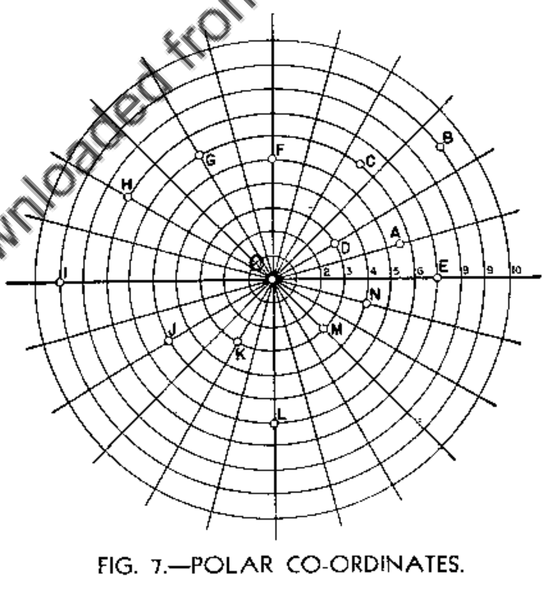

FIG. 7.—POLAR CO-ORDINATES.

3. Locate the following points:

$(\sqrt{5}, 30°)$, $\left(2\sqrt{3}, -\frac{\pi}{4}\right)$, $\left(-\frac{1}{2}\sqrt{10}, \frac{2\pi}{3}\right)$, $\left(-\sqrt{21}, -\frac{\pi}{6}\right)$, $(\sqrt{18}, 2\pi)$.

4. (a) By means of the Pythagorean theorem construct, with a selected unit, the following square roots:

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{21}$, $\sqrt{37}$, $\sqrt{40}$, $\sqrt{14}$.

(Observe that $2 = 1^2 + 1^2$, $3 = 2^2 - 1^2$, $6 = 2^2 + (\sqrt{2})^2$, etc.)

(b) Plot the following points in rectangular co-ordinates:

$(-1, -\sqrt{2})$, $(-2, \sqrt{12})$, $\left(\frac{1}{2}\sqrt{13}, 0\right)$, $(0, -\sqrt{14})$, $(5, \sqrt{20})$, $\left(\frac{1}{2}\sqrt{7}, \frac{2}{3}\sqrt{37}\right)$.

5. Find the rectangular and the polar co-ordinates of the vertices of a square with diagonal of 8 units, if the sides are parallel to the co-ordinate axes and the center of the square is at the origin.

6. Find the rectangular and the polar co-ordinates of the vertices of a square whose sides are parallel to the co-ordinate axes, if the origin lies at one vertex of the square. (4 cases.)

II.4 RELATION BETWEEN RECTANGULAR AND POLAR CO-ORDINATES

Since, as we have said, to each point in the plane there corresponds a pair of co-ordinates (x,y) and a pair (ρ,θ), we very naturally look for a relationship between the two systems. Let the polar axis of the polar system coincide with the positive x-axis,

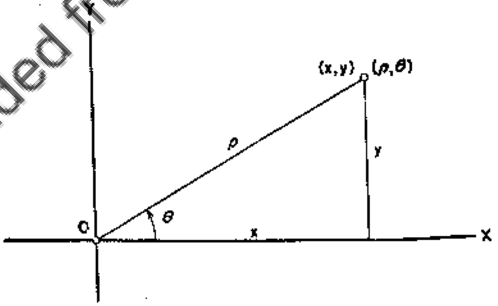

FIG. 8.—CHANGE OF CO-ORDINATES.

and the pole coincide with the origin. We can then read directly, as we see from Fig. 8,

$$(1) \qquad x = \rho \cos \theta, \quad y = \rho \sin \theta,$$

$$x^2 + y^2 = \rho^2, \qquad \frac{y}{x} = \tan \theta.$$

The student should show that these relations hold for (ρ,θ) in each of the four quadrants.

EXAMPLE 1. Write the polar co-ordinates of the point (4,4).

$$\frac{y}{x} = \frac{4}{4} = 1 = \tan 45°, \text{ therefore } \theta = 45°;$$

$$x^2 + y^2 = 16 + 16 = 32 = \rho^2, \text{ therefore } \rho = 4\sqrt{2}.$$

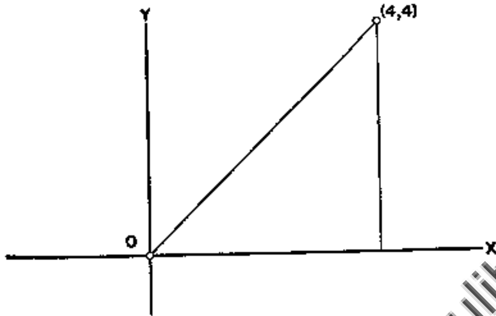

FIG. 9.—EXAMPLE 1.

Thus the point is, in polar co-ordinates, $(4\sqrt{2}, 45°)$.

EXAMPLE 2. Write the rectangular co-ordinates of the point $(6, \pi/3)$.

$$x = 6 \cos \frac{\pi}{3} = 3,$$

$$y = 6 \sin \frac{\pi}{3} = 3\sqrt{3}.$$

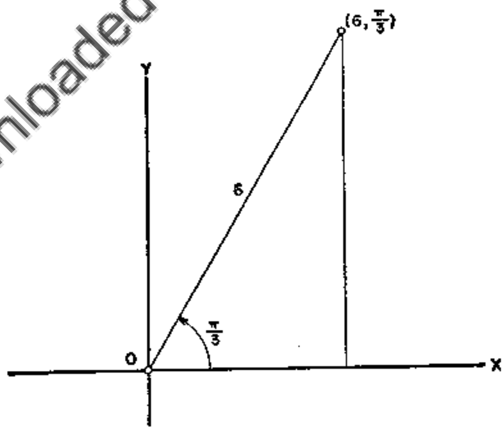

FIG. 10.—EXAMPLE 2.

Hence the point (Fig. 10) is $(3, 3\sqrt{3})$.

EXERCISES

1. Change the following co-ordinates of a set of points from rectangular to polar notation and plot the points:

   $(2,2)$, $(5,2)$, $(4,-1)$, $(-3,-2)$, $(-6,6)$, $(2,0)$, $\left(\frac{3}{2},-1\right)$, $(0,\sqrt{2})$.

2. Write the rectangular co-ordinates of the points whose polar co-ordinates are:

   $(2,45°)$, $(2,30°)$, $(4,225°)$, $(-2,750°)$, $\left(4,-\frac{2\pi}{3}\right)$, $\left(15, \text{arc tan } \frac{3}{4}\right)$, $(-8, \text{arc tan } \sqrt{15})$.

3. Through a fixed point, O, draw a pair of rectangular axes. If a particle rotates in a counterclockwise direction about O at a constant distance of 6 units from O, starting from the point $(3\sqrt{3},3)$, while the radius vector describes an angle of 195°, give its final position, using (a) polar co-ordinates, (b) rectangular co-ordinates.

   Ans. (a) $(6, 5\pi/4)$, (b) $(-3\sqrt{2}, -3\sqrt{2})$.

4. Repeat Exercise 3, with the radius vector of the particle describing an angle of 240° in the clockwise direction.

II.5 DISTANCE BETWEEN TWO POINTS

(A) Polar Co-ordinates

Let the two points be P_1 and P_2, and their co-ordinates (ρ_1,θ_1), (ρ_2,θ_2) (the subscripts on these letters are merely distinguishing tags and have no other significance). Also let d represent the distance between P_1 and P_2.

The two radii OP_1, OP_2, and the line P_1P_2 form a triangle (Fig. 11), of which the angle included by the sides OP_1 and

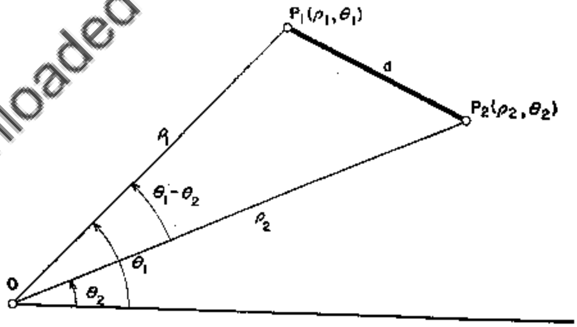

FIG. 11.—DISTANCE IN POLAR CO-ORDINATES.

OP_2 is equal to $\theta_1 - \theta_2$. The use of the Law of Cosines from trigonometry gives at once,

$$(1) \qquad d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2).$$

EXAMPLE. Find the distance between (3,75°) and (6,15°) shown in Fig. 12.

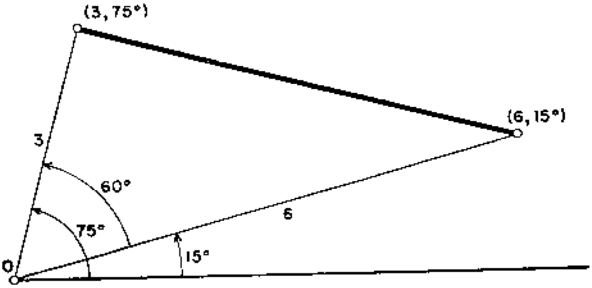

FIG. 12.

$$\rho_1 = 3,\ \rho_2 = 6,\ \theta_1 - \theta_2 = 75° - 15° = 60°.$$

Then

$$d^2 = 9 + 36 - 2 \cdot 3 \cdot 6 \cdot \cos 60° = 27.$$

(B) Rectangular Co-ordinates

The distance between two points P_1 and P_2 whose rectangular co-ordinates are (x_1,y_1), (x_2,y_2) could be found by applying the transformation from polar to rectangular co-ordinates discussed in §II.4, after making use of the addition formula for $\cos(\theta_1 - \theta_2)$. It is, however, more convenient to proceed directly, as follows:

Draw the line MP_1 perpendicular to the x-axis and P_2R parallel to the x-axis and intersecting the ordinate MP_1 in R, as shown in Fig. 13.

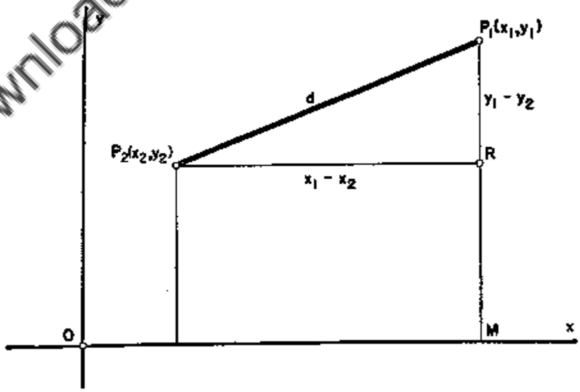

FIG. 13.—DISTANCE BETWEEN TWO POINTS.

This completes a right triangle P_2RP_1 with the required distance, d, as the length of the hypotenuse, P_1P_2. From the figure we see that $P_2R = x_1 - x_2$, and $RP_1 = y_1 - y_2$.

The Pythagorean theorem gives immediately

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2. \tag{2}$$

As an illustration of the fact that this formula applies even when the points P_1 and P_2 do not lie in the same quadrant, consider the distance between the points (3,4) and (−5, −2).

When we draw the triangle as before (Fig. 14), we note that the base has a length $8 = 3 - (-5)$, and the altitude is $6 = 4 - (-2)$. Then the distance is

$$\sqrt{8^2 + 6^2} = \sqrt{[3 - (-5)]^2 + [4 - (-2)]^2},$$

which shows that the co-ordinates of the points may be substituted directly in Formula (2).

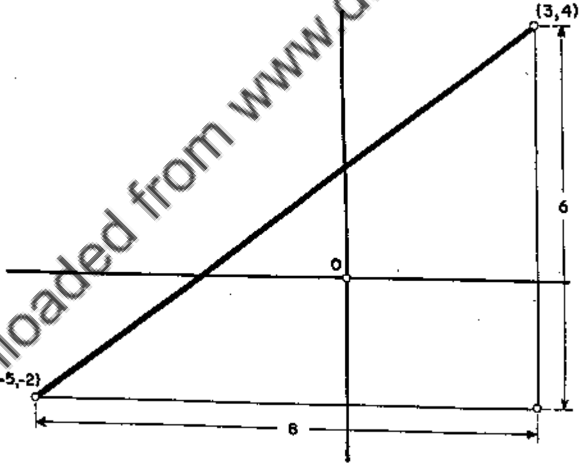

FIG. 14.

In a similar way, we can show that the formula is always valid, regardless of the positions of the points in the four quadrants.

EXERCISES

Plot the following points and find the distance between them:

1. (2,60°), (1,0°).
2. (12,135°), (16,45°).

3. $(2,60°)$, $(1,90°)$.
4. $(1,150°)$, $(5,120°)$.
5. $(-4,\pi/4)$, $(3,+\pi/4)$.
6. $(-4,\pi/4)$, $(-3,-\pi/4)$. 7. $(8,\pi)$, $(6,-\pi/2)$.
8. In Exercises 1, 3, 5, 7 change the polar to rectangular co-ordinates and then find the distance between the points.

If three or more points lie on a straight line, the points are said to be COLLINEAR.

9. By the use of the distances between points, show that the points $(12,90°)$, $(4\sqrt{3},0°)$ and $(6,30°)$ are collinear. Plot the points and draw the line.
10. Examine the following sets of points for collinearity:

 (a) $(3,3)$, $(6,2)$, $(-9,7)$. (b) $\left(\frac{1}{2},-2\right)$, $\left(-2,\frac{1}{3}\right)$, $\left(-\frac{5}{2},0\right)$.
11. Determine whether the following triangles are isosceles, equilateral or scalene:

 (a) $A(1,2)$, $B(3,4)$, $C(2,7)$.
 (b) $A(0,4)$, $B(4,1)$, $C(7,5)$.
 (c) $A(-2,0)$, $B(2,0)$, $C(0,2\sqrt{3})$.
12. Draw the quadrilateral with vertices $A(-6,-2)$, $B(5,0)$, $C(10,10)$, $D(-1,8)$. Is the figure a parallelogram? Rectangle? Rhombus? Square?

II.6 DIVISION OF A LINE SEGMENT IN A GIVEN RATIO

Consider the segment P_1P_2. Suppose it is divided by the point P, whose co-ordinates are (x,y), into two segments, whose lengths have the ratio r_1/r_2. If we draw the horizontal and vertical lines to make right triangles, as shown in Fig. 15, we have the similar

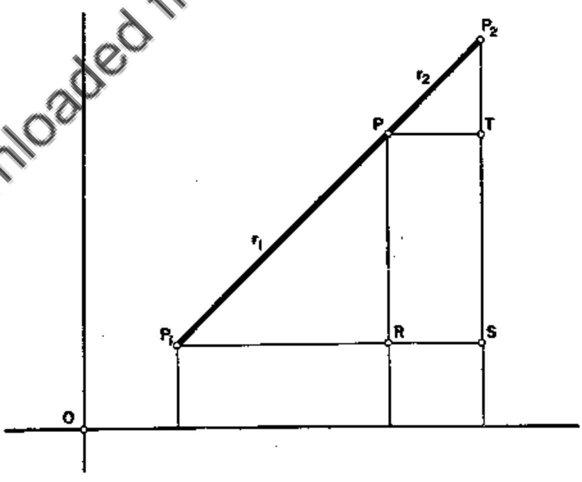

FIG. 15.—DIVISION OF LINE SEGMENT.

triangles, P_1RP and PTP_2. Setting up the proportion between sides of these similar triangles, we have

$$(1) \qquad \frac{P_1R}{PT} = \frac{x - x_1}{x_2 - x} = \frac{r_1}{r_2}, \quad \frac{RP}{TP_2} = \frac{y - y_1}{y_2 - y} = \frac{r_1}{r_2}.$$

When we solve these equations for the values of x and y we find that

$$(2) \qquad x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \quad y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}.$$

Since by the segment P_1P_2, we understand the directed segment from P_1 to P_2, the segment P_2P_1 is in the opposite direction. Hence if P_1P_2 is considered as positive, P_2P_1 would be considered negative. Then, if P lies within the segment P_1P_2, both segments P_1P and PP_2 have the same sense (either both positive or both negative) and therefore the ratio P_1P/PP_2 is positive. If the point P lies outside the segment P_1P_2 (P_2 between P_1 and P, for example), the two segments P_1P and PP_2 have opposite directions and therefore opposite signs. Then the ratio is negative.

Since division by zero is impossible, we exclude the case $r_1 + r_2 = 0$. This case implies that $P_1P = P_2P$ and there is no such point.

EXAMPLE 1. Find the trisection points of the segment (2,−1), (5,2).

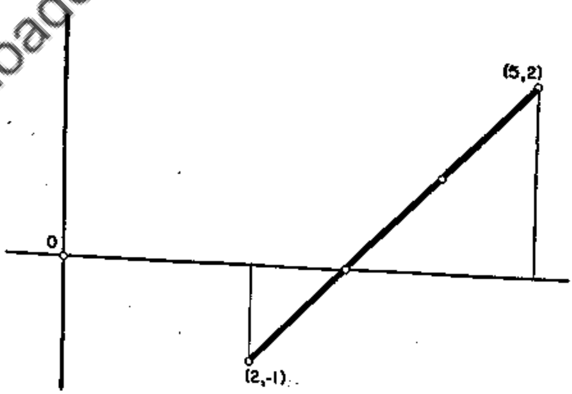

FIG. 16.—EXAMPLE 1.

There are two trisection points, which divide the segment in the ratios 1/2 and 2/1, respectively (Fig. 16). Hence for the required points we have

$$x = \frac{1\cdot 5 + 2\cdot 2}{3} = 3, \quad y = \frac{1\cdot 2 + 2\cdot(-1)}{3} = 0;$$

$$x = \frac{2\cdot 5 + 1\cdot 2}{3} = 4, \quad y = \frac{2\cdot 2 + 1\cdot(-1)}{3} = 1.$$

EXAMPLE 2. Find the points on the line (2,1), (5,2), that are twice as far from the first point as from the second (Fig. 17).

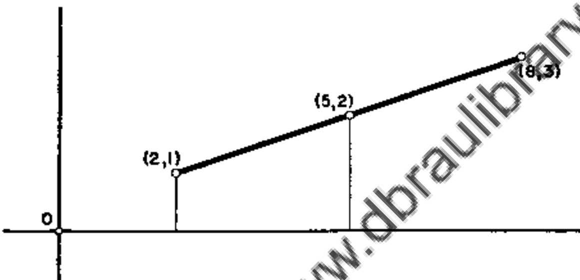

FIG. 17.—EXAMPLE 2.

Let the co-ordinates of the desired point be x and y. Since the first segment is to be twice as long as the second, it follows that the ratio must be 2/1 or $-2/1$, depending on whether (x,y) is an internal or external point of the segment. Hence the points are

$$x = \frac{2\cdot 5 + 1\cdot 2}{3} = 4, \qquad y = \frac{2\cdot 2 + 1\cdot 1}{3} = \frac{5}{3},$$

or

$$x = \frac{-2\cdot 5 + 1\cdot 2}{-1} = 8, \quad y = \frac{-2\cdot 2 + 1\cdot 1}{-1} = 3.$$

A special case of this formula, which is very important and is actually used much more often than the general formula, is that in which P is the mid-point of the segment P_1P_2. In that case, $r_1 = r_2$, and Formula (2) becomes

(3) $$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}.$$

EXERCISES

1. Solve Equations (I) of this section for x and y.
2. Find the co-ordinates of the point P, which divides the line segment P_1P_2 in the given ratio, and check the result by comparing the distances P_1P and PP_2:

   (a) $P_1(-1,-2)$, $P_2(4,3)$, ratio $= \frac{1}{4}$.

   (b) $P_1(-2,0)$, $P_2(5,1)$, ratio $= \frac{4}{3}$.

   (c) $P_1(-2,1)$, $P_2(2,3)$, ratio $= \frac{-4}{3}$.
3. Find the trisection points of the segment between the points (1,0) and (10,4).
4. Find the point on the line through (3,1) and (5,5) that is 3 times as far from the first point as from the last.
5. (a) The mid-point of a line segment is the point (2,1) and one end point of the segment is (−3,2). Find the other end point.

   (b) A segment P_1P_2 is divided in the ratio 3/2, the point of division being (0,−3). If P_1 is the point (−6,−9), find P_2.
6. Plot the triangle with vertices $A(-2,-4)$, $B(3,2)$, and $C(-6,4)$. Represent the mid-points of sides AB, BC, and CA respectively, by C', A', B'. Show that $A'B' = \frac{1}{2}BA$, $B'C' = \frac{1}{2}CB$, and $C'A' = \frac{1}{2}AC$.
7. In the triangle of Exercise 6, find the points D, E, F on the lines AA', BB', CC' (medians) that divide the medians in the ratio 2/1. Interpret the result.
8. The points $A(0,0)$, $B(a,0)$, $D(b,c)$ are vertices of a parallelogram, $ABCD$. Find the co-ordinates of C (opposite sides of a parallelogram are equal), and show by the theory of the present section, that the diagonals bisect each other.

   The center of mass for two masses is a point on the line joining the two and dividing the line into segments inversely proportional to the masses. Thus if a mass of 2 pounds is placed at A, and one of 1 pound is placed at B, the center of mass, C, divides the segment AB in the ratio ½. This is the principle of the simple lever, which we have met in physics as well as in elementary algebra.
9. (a) A mass of 3 lbs. is placed at $A(1,6)$ and a mass of 5 lbs. at $B(9,4)$. Find the center of mass, P.

   (b) Find the center of mass, Q, of a mass of 8 lbs. placed at $P(6,19/4)$, and a mass of 2 lbs. placed at $C(12,7)$. This point, Q, is the center of mass of the three masses 3,5 and 2 lbs. respectively, placed at $A(1,6)$, $B(9,4)$ and $C(12,7)$.

   If equal masses are placed at the three vertices of a triangle, the center of mass is called the **centroid** of the triangle.
10. Find the centroid of the triangle ABC of exercise 9.
11. Find the center of mass of triangle $A(-4,3)$, $B(2,-1)$, $C(0,7)$, with 3 lbs. at A, 4 lbs. at B, and 5 lbs. at C.
12. Find the centroid of triangle ABC of Exercise 11.

II.7 AREA OF A TRIANGLE

(A) Polar co-ordinates

Consider the area of the triangle (in Fig. 18) whose vertices are, the pole O, P_1,P_2. Let the two sides OP_1,OP_2 be ρ_1,ρ_2 and the in-

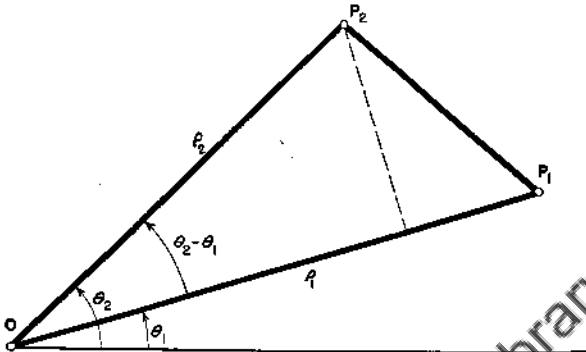

FIG. 18.—AREA OF TRIANGLE, POLAR CO-ORDINATES.

cluded angle $\theta_2 - \theta_1$. Then, from trigonometry we write

$$(1) \qquad S = \frac{1}{2}\rho_1\rho_2 \sin(\theta_2 - \theta_1),$$

$$= \frac{1}{2}\rho_1\rho_2(\sin\theta_2 \cdot \cos\theta_1 - \cos\theta_2 \cdot \sin\theta_1).$$

In order to find the area of a triangle $P_1P_2P_3$ as shown in Fig. 19, we draw the radii OP_1,OP_2,OP_3, thus forming the three triangles OP_1P_2, OP_2P_3, OP_1P_3. The desired area can be found by adding

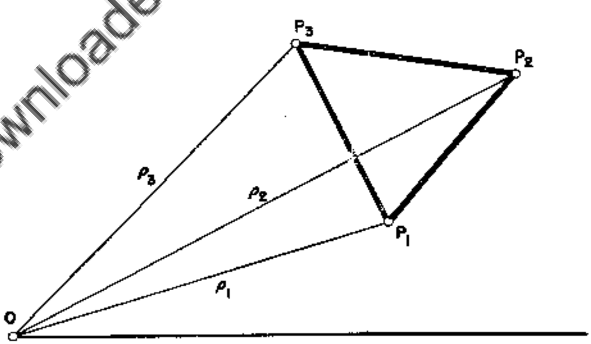

FIG. 19.—AREA OF TRIANGLE, POLAR CO-ORDINATES.

the areas of the first two triangles and subtracting from their sum the area of the third. Thus

$$(2)\quad S(P_1P_2P_3) = S(OP_1P_2) + S(OP_2P_3) - S(OP_1P_3)$$
$$= \frac{1}{2}[\rho_2\rho_1 \sin(\theta_2 - \theta_1) + \rho_3\rho_2 \sin(\theta_3 - \theta_2) - \rho_3\rho_1 \sin(\theta_3 - \theta_1)].$$

If, instead of $-\sin(\theta_3 - \theta_1)$ in the last term of (2), we write $\sin(\theta_1 - \theta_3)$, the formula may be written in the form

$$S(P_1P_2P_3) = \frac{1}{2}[\rho_1\rho_3 \sin(\theta_1 - \theta_3) + \rho_2\rho_1 \sin(\theta_2 - \theta_1) + \rho_3\rho_2 \sin(\theta_3 - \theta_2)],$$

in which the subscripts are seen to follow the cyclic order 1,2,3 as we go from one term to the next.

EXAMPLE 1. Find the area of the triangle in Fig. 20, with vertices at (4,30°), (8,45°), (2,90°).

$$S = \frac{1}{2}(8\cdot 4\cdot \sin 15° + 8\cdot 2\cdot \sin 45° - 2\cdot 4\cdot \sin 60°)$$
$$= 16\cdot 0.25882 + 8\cdot 0.70711 - 4\cdot 0.86603$$
$$= 4.14112 + 5.65688 - 3.46412$$
$$= 6.33388.$$

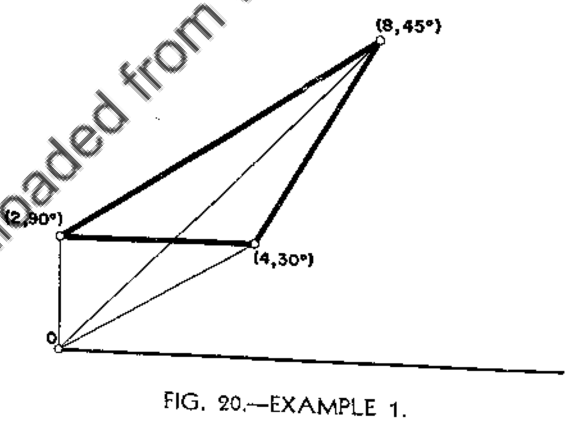

FIG. 20.—EXAMPLE 1.

(B) Area of a triangle in rectangular co-ordinates

Let the vertices of the triangle be P_1,P_2,P_3, with co-ordinates (x_1,y_1), (x_2,y_2), (x_3,y_3).

Formula (1) of §II.7(A) can be changed to rectangular co-ordinates very easily, since

(3) $\rho_1\cos\theta_1 = x_1,\ \rho_2\sin\theta_2 = y_2,\ \rho_1\sin\theta_1 = y_1,\ \rho_2\cos\theta_2 = x_2.$
Formula (1) then becomes

(4) $$S = \frac{1}{2}(x_1y_2 - x_2y_1) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

In like manner, Formula (2) reduces to

(5) $$S(P_1P_2P_3) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

For ease in memory and computation, probably (5) is the best form. However, in case the student does not know determinants of third order, he may use the equivalent form

(6) $$S = \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$
$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].$$

The student should carefully note that the points were chosen so as to describe the boundary of the triangle in the counter-clockwise manner. If the points are taken in the reverse direction, the formula gives a negative result, with a magnitude equal to the area.

EXAMPLE 2. Find the area of the triangle (2,3), (−2,5), (−4,−2).

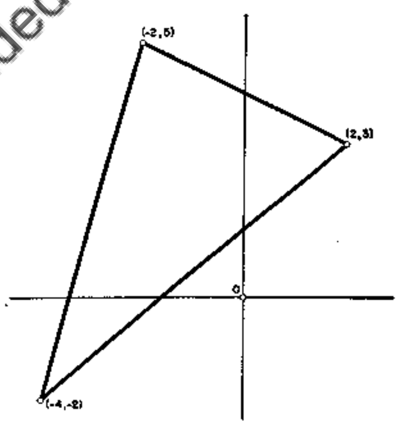

FIG. 21.—AREA OF TRIANGLE, RECTANGULAR CO-ORDINATES.

If we plot the triangle (Fig. 21), we find that the points as given are in the counterclockwise order. Then

$$S = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -2 & 5 & 1 \\ -4 & -2 & 1 \end{vmatrix} = \frac{1}{2}[2(5-(-2)) + (-2)(-2-3) + (-4)(3-5)]$$

$$= \frac{1}{2}(14 + 10 + 8) = 16.$$

II.8 AREA OF A TRIANGLE IN RECTANGULAR CO-ORDINATES (ALTERNATE PROOF)

The formula for the area of a triangle can be derived without the use of polar co-ordinates.

The area of the triangle $P_1P_2P_3$ in Fig. 22, can be found by add-

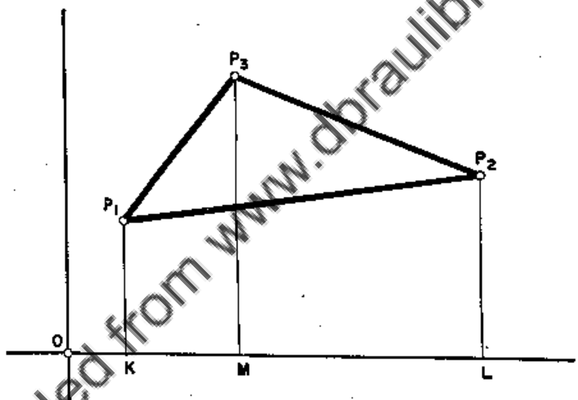

FIG. 22—AREA OF TRIANGLE, RECTANGULAR CO-ORDINATES.

ing the areas of the two trapezoids P_1KMP_3 and P_3MLP_2, and subtracting from that sum, the area of the trapezoid P_1KLP_2.

Since the area of a trapezoid is one-half the sum of the parallel sides multiplied by the distance between them, we have

(1) $$\text{Area } P_1KMP_3 = \frac{1}{2}KM(KP_1 + MP_3) = \frac{1}{2}(x_3 - x_1)(y_1 + y_3),$$

$$\text{Area } P_3MLP_2 = \frac{1}{2}ML(MP_3 + LP_2) = \frac{1}{2}(x_2 - x_3)(y_3 + y_2),$$

$$\text{Area } P_1KLP_2 = \frac{1}{2}KL(KP_1 + LP_2) = \frac{1}{2}(x_2 - x_1)(y_1 + y_2).$$

Then

(2) Area $P_1P_2P_3$ = area P_1KMP_3 + area P_3MLP_2 − area P_1KLP_2

$$= \frac{1}{2}(x_3 - x_1)(y_1 + y_3) + \frac{1}{2}(x_2 - x_3)(y_3 + y_2) - \frac{1}{2}(x_2 - x_1)(y_1 + y_2)$$

$$= \frac{1}{2}(x_3y_1 - x_1y_3 + x_2y_3 - x_3y_2 - x_2y_1 + x_1y_2)$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

EXERCISES

1. Derive Formula (5) of §II.7 from (2).
2. Find the area of the following triangles:
   (a) (1,60°), (2,90°), (3,150°).
   (b) $\left(-2, \frac{3}{4}\right)$, (5,0), $\left(4, \frac{1}{2}\right)$.
3. Change the points in Exercise 2(a), to rectangular co-ordinates, compute the area in the new system, and thus check the result.
4. If three points lie on a straight line, what is the area of the triangle? If the area of the triangle is zero, can you state any property of the three vertices? Use the results in testing whether the following sets of points are collinear.
   (a) (0,6), (8,0), (4,3).
   (b) (1,5), (7,−2), (3,10).
   (c) (x_1,y_1), (x_2,y_2), $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
5. Given the triangle $A(-4,-3)$, $B(4,3)$, $C(0,5)$. Let A', B', C' be the midpoints of the sides BC, CA and AB respectively. Find the ratio of the areas of triangles $A'B'C'$ and ABC.
6. Given the triangle $(a,0)$, $(0,b)$, $(-c,0)$. Show that the triangle whose vertices are the midpoints of the sides of the given triangle has an area ¼ that of the original triangle.
   Note that by choosing the points as we have, the given triangle may be any triangle, and hence that Exercise 6 proves a general property of all triangles.
7. Show from either (5) or (6) of §II.7 that if the vertices of the triangle are taken in a clockwise order, the result obtained for the area has a sign opposite to that found with the points taken in counterclockwise order.
8. Find the length of the altitude of triangle $A(4,2)$, $B(0,6)$, $C(-5,4)$ drawn from B.
9. Plot the triangle whose vertices are at $P(1,-5)$, the origin, and $Q(4,0)$. Find the area.
10. The vertices of a triangle are $P(-2,1)$, $Q(1,y)$, $R(4,9)$. The area is three square units. Find y (two possible answers).

II.9 SLOPE OF A LINE

Suppose a line meets the x-axis at an angle α. This angle is the one above the x-axis and to the right of the line (see Fig. 23). It is called the **angle of inclination** of the line. We find it convenient to have a numerical measure of the steepness of ascent of the line, as we go along the line, say, from left to right. If the angle of inclination is more than 90°, the ascent becomes a descent, and therefore the numerical measure should be negative. We find that the tangent of the angle of inclination serves the purpose very well. We call this measure the **slope** of the line and quite generally designate it by m.

The slope of a line can be expressed directly in terms of the angle of inclination, that is,

$$m = \tan \alpha. \tag{1}$$

The numerical value can also be found in terms of the coordinates of two distinct points on the line. Let the two points on the line be P_1, P_2, as shown in Fig. 23. Draw the horizontal

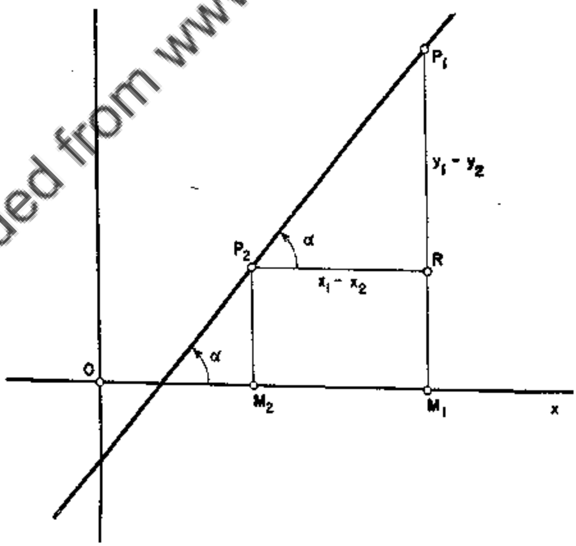

FIG. 23.—SLOPE OF LINE.

and vertical lines forming the right triangle P_2RP_1. From trigonometry we recall that

(2) $$m = \tan \alpha = \frac{RP_1}{P_2R} = \frac{y_1 - y_2}{x_1 - x_2}.$$

If the angle α is zero, the line is horizontal, and the slope is zero. For every value of α there is a single value of $\tan \alpha =$ slope, except for $\alpha = 90°$. For this particular value (when the line is vertical), the value of the tangent does not exist, since division by zero is excluded from our algebra. However, we have been accustomed to indicating this case by the symbol ∞ and to saying that the tangent of 90° is infinite. Inasmuch as this can introduce no misunderstanding, we shall adopt the policy of calling the slope of a vertical line infinite, though a very strict interpretation would demand that we always say **it does not exist.** In most cases a figure will suggest the procedure to be followed in situations involving this special case.

It is suggested that the student actually construct figures and work out the details to show that the form of (2) is always the same regardless of the quadrants in which P_1 and P_2 fall.

EXERCISES

1. Draw the line through each of the following pairs of points. Find its slope, m, and the angle of inclination, α.

   (a) (5,2), (10,7).
   (b) (−6,0), (2,8).
   (c) (−1,5), (0,4).
   (d) (6,6), (−2,0).
   (e) (0,−7), (8,2).
   (f) (−3,4), (5,4).
   (g) (4,−3), (4,5).

2. Draw the triangle whose vertices are $A(-3,0)$, $B(5,-1)$, $C(3,8)$. Find the slopes of the sides.
3. Draw the lines given by (4,7), (5,9) and (−3,0), (−2,2) on the same set of co-ordinate axes. Are they parallel? Why?
4. Are the lines 1(a), 1(c) perpendicular? Why? Are the lines determined by (2,7), (5,9) and (6,6), (8,3) perpendicular? What is the relation between their slopes?
5. Draw any triangle with vertices $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$. Let A', B' be the midpoints of the sides BC, CA. Show that $B'A'$ has the same slope as AB and is half as long as AB. What is the theorem in geometry that is thus proved?
6. By means of slopes, determine whether the following sets of points are collinear:

   (a) (1,1), (−6,−4), (4,2).
   (b) (2,2), (0,−5), (−2,8).
   (c) (3,4), (5,6), (−1,0), (0,−5).

SUMMARY OF CHAPTER II

Relations between rectangular and polar co-ordinates

$$x = \rho \cos \theta \qquad x^2 + y^2 = \rho^2$$

$$y = \rho \sin \theta \qquad \frac{y}{x} = \tan \theta$$

Distance between two points

$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos (\theta_1 - \theta_2)$$
$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

Division of a line segment in given ratio r_1/r_2

$$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2} \qquad y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}$$

Special case: Mid-point of segment

$$x = \frac{x_1 + x_2}{2} \qquad y = \frac{y_1 + y_2}{2}$$

Area of triangle $P_1P_2P_3$

$$S = \frac{1}{2}[\rho_2\rho_1 \sin (\theta_2 - \theta_1) + \rho_3\rho_2 \sin (\theta_3 - \theta_2) - \rho_3\rho_1 \sin (\theta_3 - \theta_1)]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Slope of a line

$$m = \tan \alpha = \frac{y_1 - y_2}{x_1 - x_2}$$

CHAPTER

III

THE STRAIGHT LINE

III.1 LOCI

As we may recall from plane geometry, a **locus** consists of all points that satisfy a given condition, and no other points. Thus, the locus of points equidistant from two fixed points is the perpendicular bisector of the line segment joining them. The locus of points equidistant from two intersecting lines is the bisectors of the angles between the lines. The locus of points at a constant distance from a given point is a circle.

As we saw in the beginning of Chapter II, we are concerned with the twofold problem of finding the geometric locus described by an equation in which the co-ordinates of a point are the variables, and of determining the equation that will describe a given locus. For the sake of simplicity, we limit our variables, as before, to polar and rectangular co-ordinates.

An important phase of our work, so far as experimental science is concerned, is the establishment of an equation that will describe the results of experiment. Since the attack on this problem involves a determination of the nature of the locus as well as the setting up of the desired equation, we can take it up only after the rest of our study is pretty well completed.

We shall begin our study of loci with the straight line.

III.2 THE STRAIGHT LINE IN POLAR CO-ORDINATES

Suppose we have a straight line perpendicular to the radius vector of the point (p,ω) and passing through that point (Fig. 24).

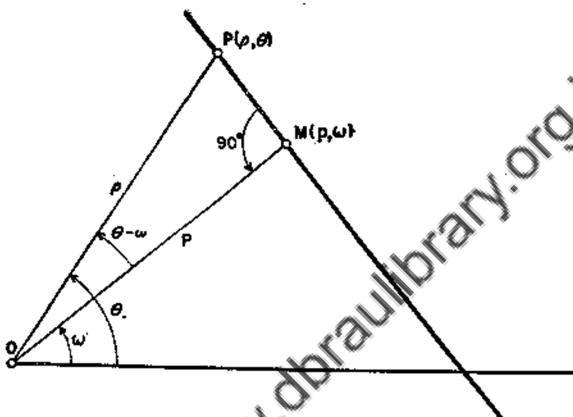

FIG. 24.—STRAIGHT LINE IN POLAR CO-ORDINATES.

Let M be the given point, and P be any point of the line. The triangle OMP is a right triangle, with the angle at the origin equal to $\theta - \omega$. Then, from trigonometry,

(1) $$\rho \cos (\theta - \omega) = p,$$

which is the equation in polar co-ordinates (ρ and θ) of the line shown.

EXERCISES

In each of the following, write the equation of the line determined by the given data.

1. $p = 3, \omega = 45°$.
2. $p = 2, \omega = 60°$.
3. $p = 1, \omega = 120°$.
4. $p = 4, \omega = 135°$.
5. $p = 6, \omega = \frac{\pi}{2}$.
6. $p = 3, \omega = -30°$.
7. $p = 2, \omega = 0°$.
8. $p = 4, \omega = \pi$.
9. $p = 0, \omega = 45°$.

In each of the following cases, examine whether the given point lies on the given line.

10. $\left(8,\frac{\pi}{2}\right)$, $\rho \cos (\theta - 30°) = 4$.
11. $(7,30°)$, $\rho \cos (\theta - 30°) = 8$.
12. $(4,120°)$, $\rho \cos (\theta - 60°) = 2$.
13. $\left(-6,\frac{\pi}{6}\right)$, $\rho \cos (\theta - 150°) = 3$.
14. $(4,\pi)$, $\rho \cos (\theta - 45°) = 2\sqrt{2}$.

III.3 NORMAL FORM OF THE EQUATION OF A STRAIGHT LINE

The radius vector p (see Fig. 24, p. 28) is said to be **normal** to the line MP. (**Normal** and **perpendicular** are sometimes used interchangeably.)

If we apply the transformation from polar to rectangular co-ordinates, Equation (1) of §II.4,

(1) $$\rho \cos(\theta - \omega) - p = 0,$$

or $$\rho \cos \theta \cos \omega + \rho \sin \theta \sin \omega - p = 0,$$

becomes, since $\rho \cos \theta = x$ and $\rho \sin \theta = y$,

(2) $$x \cos \omega + y \sin \omega - p = 0.$$

Every straight line has an equation that may be written in this form, since, clearly, through the pole or origin, there can always be drawn the normal to the line, whose length will be p (p may, of course, be zero), and this normal will always make a definite angle, ω, with the positive x-axis. Since this equation is of the first degree in x and y, it follows that:

Every straight line has an equation of the first degree in x and y.

One question may occur to the thoughtful student. Does every equation of the first degree in x and y represent a straight line? The answer is yes, as the following discussion shows.

Consider the most general form of the equation of the first degree in x and y,

(3) $Ax + By + C = 0$, where A,B,C are constants.

Here, of course, if the equation is to be of the first degree, A and B cannot both be zero.

If Equation (3) can be reduced to the form (2) we shall know that it represents a straight line. A and B may be greater than unity, while $\sin \omega$ and $\cos \omega$ are never greater than unity in absolute value. Suppose we attempt to fit Equation (3) to the form (2) by dividing by a constant, k. The resulting equation is

(4) $$\frac{A}{k}x + \frac{B}{k}y + \frac{C}{k} = 0.$$

If this equation is to be of the form (2), we must have

(5) $$\cos\omega = \frac{A}{k}, \quad \sin\omega = \frac{B}{k}, \quad p = -\frac{C}{k}.$$

We remember that

(6) $$\sin^2\omega + \cos^2\omega = 1,$$

is an identity that holds for all angles. Substitution of the values of $\cos\omega$ and $\sin\omega$ from (5) in (6) gives

(7) $$\left(\frac{A}{k}\right)^2 + \left(\frac{B}{k}\right)^2 = 1,$$

and consequently

(8) $$k = \pm\sqrt{A^2 + B^2}.$$

There remains the question as to the proper sign before the radical. If we assume that p is positive, C/k must be negative. Then we must use the sign opposite to that of the constant term. In case the constant term is zero, we may assume that $\omega < 180°$. Then $B/k = \sin\omega$ must be positive, and therefore we use the same sign as that of the y-term.

We have shown that every equation of the form (3), the most general equation of the first degree in x and y, can be reduced to the form (2), which we know to be the equation of a straight line. It follows that

Every equation of the first degree in x and y represents a straight line.

EXAMPLE 1. Reduce the equation $5x - 12y - 26 = 0$ to the normal form.

$$A = 5,\ B = -12,\ C = -26.\ k = \pm\sqrt{5^2 + 12^2} = \pm 13.$$

Since $C = -26$, we choose the + sign, and write

$$\frac{5}{13}x - \frac{12}{13}y - 2 = 0.$$

Hence

$$\cos \omega = \frac{5}{13}, \quad \sin \omega = -\frac{12}{13}, \quad p = 2.$$

EXAMPLE 2. Write the normal form of the equation of the straight line for which $p = 4$, $\omega = -45°$.

Direct substitution for p and ω in (2) gives at once

$$\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y - 4 = 0.$$

EXERCISES

1. Write the normal form of the equation of the straight line for which
   (a) $\omega = 45°$, $p = 4$.
   (b) $\omega = 135°$, $p = 2$.
   (c) $\omega = 0°$, $p = 5$.
   (d) $\omega = 90°$, $p = 1$.
   (e) $\omega = 30°$, $p = 6$.
   (f) $\omega = 60°$, $p = 2$.
   (g) $\omega = \text{arc tan}\left(\frac{5}{12}\right)$, $p = 3$.
   (h) $\omega = \text{arc tan}\left(\frac{3}{4}\right)$, $p = 2$.
2. Reduce the following equations to normal form and determine the values of ω, and p:
   (a) $3x - 4y - 15 = 0$.
   (b) $12x + 5y + 26 = 0$.
   (c) $7x - 24y + 50 = 0$.
   (d) $x + y - 6 = 0$.
   (e) $x\sqrt{3} + y - 10 = 0$.
   (f) $x - y = 0$.
   (g) $2x + 3y + 4 = 0$.
   (h) $6x + 4y - 9 = 0$.
3. Draw the straight lines defined in Exercise 1.
4. Draw the straight lines corresponding to the equations in Exercise 2.
5. Write the polar equations of the lines defined in Exercise 1.

   Ans. 1. (a) $\rho \cos\left(\theta - \frac{\pi}{4}\right) = 4$. (e) $\rho \cos\left(\theta - \frac{\pi}{6}\right) = 6$.
   (c) $\rho \cos \theta = 5$.
6. Write the polar equations of the lines corresponding to the equations in Exercise 2.

   Ans. 2. (a) $\rho \cos\left[\theta - \text{arc tan}\left(-\frac{4}{3}\right)\right] = 3$. (f) $\theta = \frac{\pi}{4}$.
   (d) $\rho \cos\left(\theta - \frac{\pi}{4}\right) = 3\sqrt{2}$.
7. Derive Equation (2) assuming that ω is in the third quadrant.
8. Show that the form of (2) is not changed if ω is replaced by $\omega + 180°$ and p by $-p$.

III.4 DISTANCE OF A POINT FROM A LINE

Suppose we have the equation of the line given in normal form,

$$(1) \qquad x \cos \omega + y \sin \omega - p = 0.$$

If the equation is not already in normal form, we can make it so by applying the method of the preceding section (§III.3).

In Fig. 25, $OM = p$.

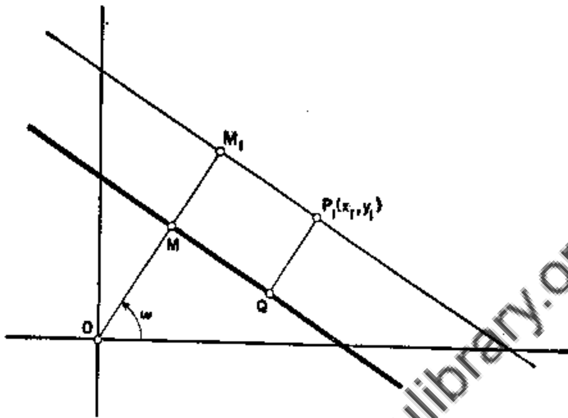

FIG. 25.—DISTANCE OF A POINT FROM A LINE.

Pass a second line through the given point (x_1,y_1) parallel to (1). Since the normal to the second line makes with positive x-axis the angle ω or $\omega + 180°$ accordingly as the two lines are on the same or opposite sides of the origin, the only difference in the equations is that p is replaced by $p_1(=OM_1)$, so that the second line has the equation

(2) $$x \cos \omega + y \sin \omega - p_1 = 0,$$

where p_1 is positive or negative accordingly as the line is on the same or opposite side of the origin as (1), and is zero if the line passes through the origin.

Since the line (2) passes through the point P_1, the co-ordinates (x_1,y_1) satisfy Equation (2), and hence

(3) $$x_1 \cos \omega + y_1 \sin \omega - p_1 = 0.$$

The required distance, QP_1, is seen to equal MM_1, hence

(4) $$QP_1 = p_1 - p = x_1 \cos \omega + y_1 \sin \omega - p.$$

Since p is either positive or zero, it follows that if QP_1 (the directed segment, reading from the line to the point) in (4) is positive, P_1 and the origin are on opposite sides of the line (1). If QP_1 is negative, they are on the same side of the line.

EXAMPLE. Find the distance from the line $24x - 7y + 50 = 0$ to the point (1,4), shown in Fig. 26.

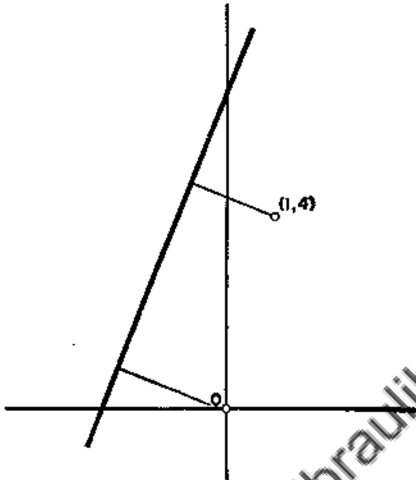

FIG. 26.

Reducing the equation to the normal form,

$$-\frac{24}{25}x + \frac{7}{25}y - 2 = 0,$$

and then, substituting the co-ordinates (1,4), we have

$$d = \left(\frac{-24 + 28}{25}\right) - 2 = -\frac{46}{25}.$$

EXERCISES

1. In each of the following, find the distance of the indicated point from the given line:
   (a) $3x + 4y + 5 = 0$, (2,5).
   (b) $x + 7y - 19 = 0$, (3,1).
   (c) $5x + 4y - 24 = 0$, (4,1).
   (d) $2x - y + 2 = 0$, (2,−2).
   (e) $x + y = 6$, (12,−6).
2. Set up the condition that the point (x,y) shall be equally distant from the two lines,
   $$5x + 12y - 10 = 0,\ 7x - 24y + 14 = 0.$$
   May there be two answers?
3. Find the equations of the bisectors of the angles between the lines,
   $$2x + 3y - 6 = 0,\ 3x - 2y - 6 = 0.$$
4. (a) Show that the line $2x + 3y = 7$ passes through (2,1), (5,−1).
   (b) Find the distance from the line $2x + 3y - 7 = 0$ to (3,2).
   (c) Find the distance between (2,1) and (5,−1).
   (d) Using the results of (b) and (c), find the area of the triangle whose vertices are (2,1), (5,−1), (3,2).
   (e) Verify this result by use of the formula in §II.7.

5. (a) Show that the points (x_1,y_1), (x_2,y_2) are both on the line

$$(y_1 - y_2)x + (x_2 - x_1)y + (x_1y_2 - x_2y_1) = 0.$$

(b) Find the distance between (x_1,y_1) and (x_2,y_2).

(c) Find the distance of the point (x_3,y_3) from the line of Exercise 5(a).

(d) Using the results of (b) and (c), find the area of the triangle

$$(x_1,y_1),\ (x_2,y_2),\ (x_3,y_3).$$

(e) Check the result of (d) against the formula in §II.7.

6. Find the equations of the bisectors of the interior angles of triangle $A(2,-1)$, $B(5,1)$, $C(-1,10)$, and show that these three lines are concurrent.

7. Repeat Exercise 6, using the triangle (x_1,y_1), (x_2,y_2), (x_3,y_3), thus proving the theorem for the general case. State the theorem.

8. By the method of §III.3 draw the lines

(1) $3x - 4y + 6 = 0$,
(2) $5x - 12y - 6 = 0$,
(3) $3x + 4y - 18 = 0$,

and find the points of intersection A, B, C, of the pairs (1),(2); (2),(3);(3),(1).

9. Find the equation of the bisector of angle C of the triangle ABC in Exercise 8, and its intersection, C', with the side AB.

III.5 STRAIGHT LINE THROUGH TWO POINTS

A straight line is determined by any two of its points. Given the two points (x_1,y_1), (x_2,y_2), the equation of the straight line may be written in several ways.

(A) Determinant method

If the point $P(x,y)$ lies on the line P_1P_2, the area of the triangle PP_1P_2 must be zero. Then, from the area formula,

(1)
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Expanding this determinant, we may write

(2)
$$(y_1 - y_2)x - (x_1 - x_2)y + (x_1y_2 - x_2y_1) = 0.$$

(B) Slope method

If the point P (Fig. 27) lies on the line P_1P_2, the slope of PP_1 is the same as the slope of P_1P_2. Hence

(3)
$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}. \quad (x_1 \neq x_2)$$

If $x_1 = x_2$, the line is perpendicular to the x-axis and its equation is $x = x_1$ (or x_2).

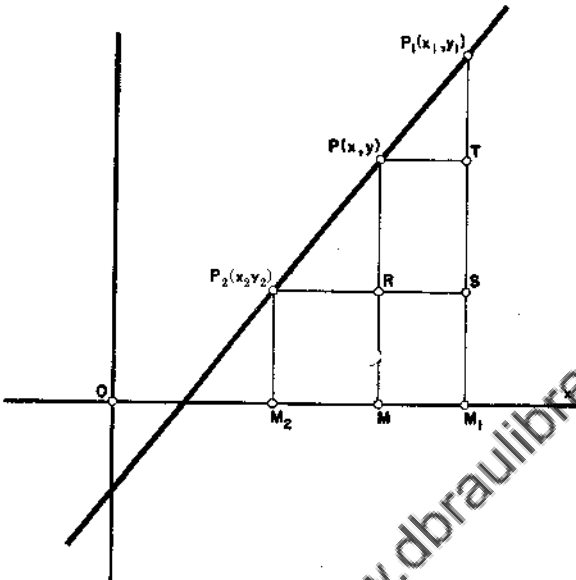

FIG. 27.—LINE THROUGH TWO POINTS.

If we clear of fractions and collect terms in Equation (3), we get Equation (2).

EXAMPLE 1. Write the equation of the straight line (Fig. 28) through the two points (5,2) and (7,−3).

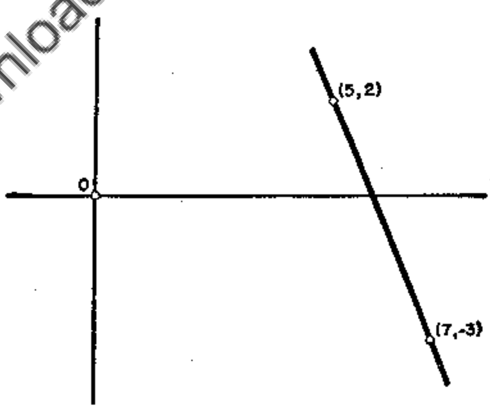

FIG. 28.

If we use Equation (1),

$$\begin{vmatrix} x & y & 1 \\ 5 & 2 & 1 \\ 7 & -3 & 1 \end{vmatrix} = 0 = 5x + 2y - 29.$$

EXAMPLE 2. Write the equation of the straight line (Fig. 29) through (1,4), (−2,−1).

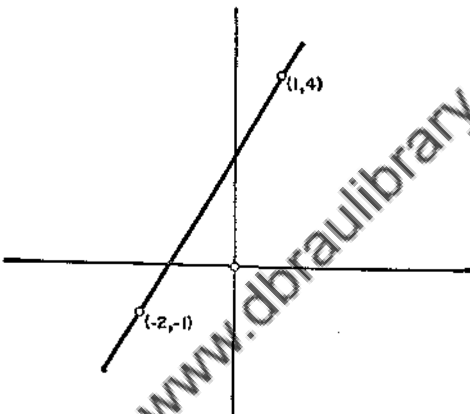

FIG. 29.

If we use Equation (3),

$$\frac{y-4}{x-1} = \frac{4-(-1)}{1-(-2)}, \text{ or } 3y - 12 = 5x - 5,$$

which reduces to

$$5x - 3y + 7 = 0.$$

EXERCISES

Write the equations of the straight lines through the following pairs of points and reduce the equation in each case to the form $Ax + By + C = 0$.

1. (2,5), (3,−2).
2. (6,3), (2,1).
3. (−1,2), (8,−4).
4. (−2,−2), (−5,−2).
5. (−1,−7), (−1,2).
6. $(a,-b)$, $(-a,b)$.

Reduce the following equations to the normal form.

7. $4x - 3y + 25 = 0$.
8. $12x + 5y = 39$.
9. $y = 4x - 3$.
10. $x - y = 2\sqrt{2}$.
11. $2x - 3y = 5$.
12. $y = \frac{7}{24}x + 2$.
13. Find the equations of the sides of the triangle whose vertices are $A(0,-4)$, $B(5,2)$, $C(-1,6)$, and check by solving the resulting equations in pairs.

14. (a) Find the equation of the line through (1,1) and (−6,−4) and show that the point (8,6) lies on this line.
(b) By applying the theory of this section, examine whether the points (2,2), (0,5), (−2,8), and (4,−1) are collinear.

15. Plot the quadrilateral with vertices $A(-4,0)$, $B(6,0)$, $C(5,4)$, $D(0,4)$.
(a) Show that DC is parallel to AB.
(b) Find E, the point of intersection of AD and BC.
(c) Letting M be the mid-point of DC, show that EM passes through the mid-point of AB.

16. Find a point of the line $4x - 5y + 40 = 0$ such that the distance from the line through (0,3) and (4,0) to that point shall be +10.2.

17. (a) Plot the triangle $A(2,-1)$, $B(5,1)$, $C(-1,10)$, and show that the medians are concurrent.
(b) Repeat 17(a), using (x_1,y_1), (x_2,y_2), (x_3,y_3), thus proving the general theorem that the medians of a triangle are concurrent.

18. Show that the form of Equation (3) of III.5 is the same regardless of the quadrants in which $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ fall.

III.6 SLOPE OF A LINE

In Equation (3) of the preceding section, §III.5, we note that the slope of the line defined by (x_1,y_1) and (x_2,y_2) is

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

This is the negative of the quotient of the x-coefficient by the y-coefficient in Equation (2) of §III.5. It can be shown that for the general straight line, whose equation is

$$Ax + By + C = 0, \tag{1}$$

this same property holds. That is to say, the slope of the line (1) is $-A/B$, or minus the quotient of the coefficient of x divided by the coefficient of y. If the y-coefficient, B, is zero, the line is perpendicular to the x-axis.

EXAMPLE. Show, by means of their slopes, that the lines

$$3x - 2y = 9,\ 6x - 4y + 7 = 0,$$

are parallel.

The slope of any line, $Ax + By + C = 0$, is $-A/B$; hence the slopes of the given lines are, respectively, 3/2 and 6/4 = 3/2. Since the slopes are the same, the angles of inclination are the same and therefore the lines are parallel (Fig. 30, page 38).

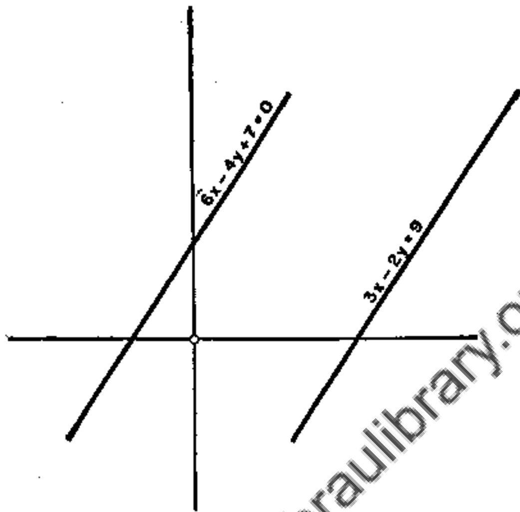

FIG. 30.—PARALLEL LINES.

EXERCISES

Determine the slope of each of the following straight lines:

1. $2x + y - 9 = 0$.
2. $4x - 2y - 17 = 0$.
3. $5x + 2y - 10 = 0$.
4. $x - y = 1$.
5. $2x + 2y - 5 = 0$.
6. $6x - 19 = 0$.
7. $4y - 5 = 0$.
8. $(m - n)x - (m^2 - n^2)y + 1 = 0$.
9. Given the straight line $Ax + By + C = 0$, where $B \neq 0$.
   Find the points whose abscissas are x_1 and x_2, respectively.
   Find the slope of the line joining these two points and show that it is equal to $-A/B$.

III.7 GRAPHING A STRAIGHT LINE

The simplest method of graphing a straight line is to find the co-ordinates of two points on the line, plot them, and use a straightedge.

In general it is easiest to find the points where the line intersects the axes. For example, $2x + 3y = 6$ cuts the axes in (3,0), (0,2).

It may happen that the co-ordinates of these points are fractions, and thus somewhat troublesome to plot accurately. In that event, it may be possible to determine, by inspection, some point whose co-ordinates are integers.

For instance, the line $2x + 3y - 7 = 0$ has intercepts (distance from the origin to the intersections with the axes) $7/2$ and $7/3$, and these are not too easy to measure. However, the point (2,1) lies on this line. The slope of the line is $-2/3$. If we start from point (2,1) and go 3 units to the right and 2 units down, we shall have another point on the line, (5,−1).

In general, having found one point on a line with slope a/b, we can find a second point by moving b units to the right and a units up (a negative numerical value for a or b merely reverses the direction of motion).

EXAMPLE. Plot the graph of the equation

$$5x + 12y - 7 = 0.$$

The line passes through (−1,1) and has a slope of $-5/12$. We may pass 12 units to the right and 5 units down, thus finding (11,−4) as a second

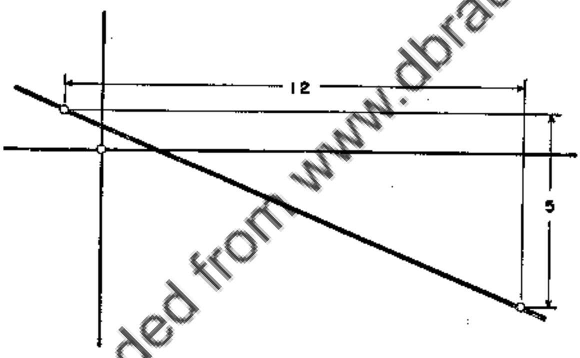

FIG. 31.—GRAPH OF $5x + 12y - 7 = 0$.

point on the line (Fig. 31). In actual practice, one does not usually note the numerical values of the co-ordinates of the second point, and the work is quite rapid.

EXERCISES

Plot the straight lines whose equations are:

1. $4x + 5y - 20 = 0$.
2. $5x - y = 7$.
3. $2x + y = 5$.
4. $3x - 2y = 15$.
5. $4x + 7y + 3 = 0$.

Graph the following straight lines:

6. Through (1,−1) with slope 2.
7. Through (5,−2) with slope $\frac{3}{4}$.
8. Through (0,5) with slope $-\frac{1}{2}$.
9. Through (2,−1) with slope 0.
10. Through (−2,5) with slope $-\frac{5}{2}$.

III.8 THE POINT-SLOPE FORM OF THE EQUATION OF THE STRAIGHT LINE

A straight line is uniquely determined when a point on the line and its direction, or slope, are given.

If the given straight line is to pass through a fixed point $P_1(x_1,y_1)$ and have a given slope, m, we must have,

(1) $$m = \frac{y - y_1}{x - x_1},$$

where (x,y) is any point on the given line.

This equation is more frequently found in the form without fractions,

(2) $$y - y_1 = m(x - x_1).$$

If the student is interested in writing such an equation rapidly, and in its simplest form, he might observe that we have, by rearranging terms,

(3) $$mx - y - (mx_1 - y_1) = 0.$$

Thus we may write the equation of the line (Fig. 32) through (4,5) having the slope 2:

$$2x - y - (2\cdot 4 - 5) = 0$$

or

$$2x - y - 3 = 0.$$

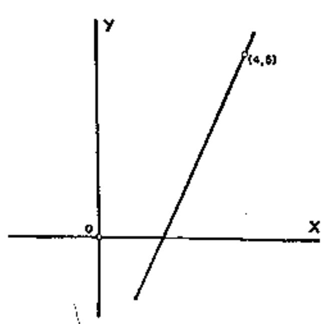

FIG. 32.—LINE THROUGH (4,5) WITH SLOPE 2.

In practice, the last equation would be the only one written, the rest of the work being done mentally.

Suppose the slope to be a fraction, $m = p/q$. Then we should have

$$\frac{p}{q}x - y - \left(\frac{p}{q}x_1 - y_1\right) = 0. \tag{4}$$

It is more convenient if we clear of fractions, getting

$$px - qy - (px_1 - qy_1) = 0. \tag{5}$$

EXAMPLE. Write the equation of the line (Fig. 33) through (6,5) with slope $-4/3$.

$$4x + 3y - (4\cdot 6 + 3\cdot 5) = 0.$$
$$4x + 3y - 39 = 0.$$

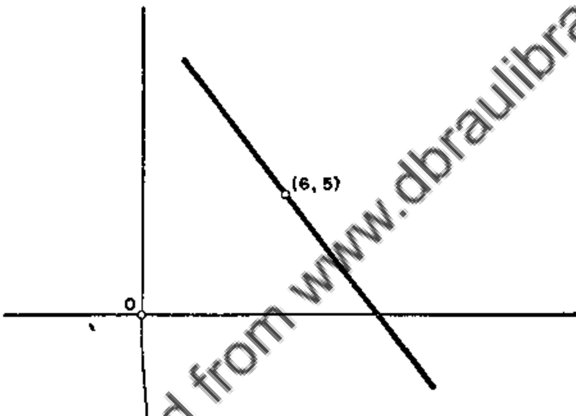

FIG. 33.—GRAPH OF $4x + 3y - 39 = 0$.

EXERCISES

In each of the following, write the equation of the line with the given slope through the indicated point:

1. (3,4), $m = 2$.
2. (1,−2), $m = 7$.
3. (2,−3), $m = -1$.
4. (5,6), $m = -\frac{1}{3}$.
5. (7,−3), $m = -2$.
6. (−1,−2), $m = \frac{1}{2}$.
7. (−3,−2), $m = \frac{2}{3}$.
8. (−7,2), $m = -\frac{3}{4}$.
9. Plot the line through (2,5) with slope $m = -\frac{3}{4}$. Find the distance of the point (0,6) from this line.
10. Find the distance of the point (12,−2) from the line whose x-intercept is 9, and whose slope is $\frac{1}{3}$.
11. (a) Given the point $A(b,c)$ in the first quadrant, and points $B(0,0)$ $C(a,0)$ (a positive). Find the coordinates of a point D, such that $ABCD$ shall be a parallelogram.
    (b) If E be the mid-point of AD, show that BD is divided in the ratio 2/1 by the point of intersection of CE and BD.

III.9 INTERCEPT FORMS OF THE EQUATION OF A STRAIGHT LINE

Suppose the straight line to have the slope m, and to pass through the point $(0,b)$, i.e. have the y-intercept b. (Fig. 34.)

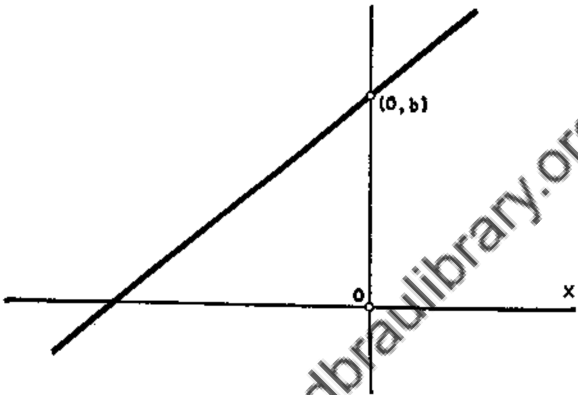

FIG. 34.—LINE WITH FIXED y-INTERCEPT.

The equation, according to the preceding §III.8, is

(1) $$mx - y + b = 0,$$

or, solving for y,

(2) $$y = mx + b.$$

This form, known as the **y-intercept-slope form,** is particularly useful in solving equations by the substitution method. The equation of any straight line, if it contains a y-term, can be reduced to this form by solving for y in terms of x. Then the slope and y-intercept can be read directly.

Another form, known as the **intercept form** for the equation of the straight line, involves both the x- and y-intercepts. Suppose the line to have the intercepts a and b, that is, to pass through the two points $(a,0)$, $(0,b)$. We can readily see that the slope of this line is $-b/a$, and hence the equation is,

(3) $$bx + ay - ab = 0.$$

Transposing the constant term and dividing by ab, we have the standard form,

(4) $$\frac{x}{a} + \frac{y}{b} = 1.$$

EXAMPLE. The equation of the line with slope 3 and y-intercept 5, is

$$y = 3x + 5.$$

The line with x- and y-intercepts 2 and 5 has the equation

$$\frac{x}{2} + \frac{y}{5} = 1.$$

EXERCISES

1. Reduce the following equations to the y-intercept-slope form:
   (a) $4x + y - 5 = 0$. **Ans.** $y = -4x + 5$.
   (b) $x - 3y + 4 = 0$.
   (c) $5x + 7y - 12 = 0$.
   (d) $Ax + By + C = 0$.
2. Write the intercept form of the following lines with the intercepts:
   (a) 2, 3. (b) $\frac{1}{2}$, 4.
3. Write the equations of the following lines with the given slope and y-intercept:
   (a) $m = 3, b = 6$. (c) $m = 5, b = 0$.
   (b) $m = 1, b = -2$. (d) $m = -1, b = 3$.
   (e) $m = -2, b = -1$.
4. Reduce the following equations to the intercept form and read the values of the intercepts:
   (a) $4x - 3y - 12 = 0$. (c) $5x - 2y + 13 = 0$.
   (b) $2x + y - 6 = 0$. (d) $Ax + By + C = 0$.

III.10 ANGLE BETWEEN TWO LINES

The angle that a line, l_1, makes with a line, l_2, is the positive angle through which one must rotate l_2 in order to make it parallel to (or coincident with) l_1.

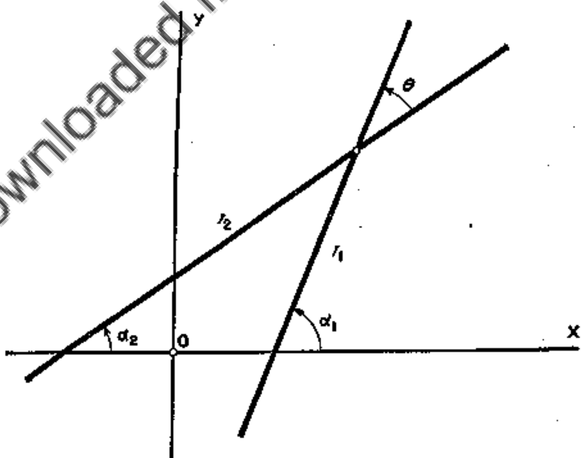

FIG. 35.—ANGLE BETWEEN TWO LINES.

If the two lines have direction angles α_1 and α_2 (see Fig. 35), we see that the angle that l_1 makes with l_2 is

(1) $$\theta = \alpha_1 - \alpha_2.$$

Then

(2) $$\tan\theta = \tan(\alpha_1 - \alpha_2), = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}, = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

If $\tan\theta$ is negative, the angle that l_1 makes with l_2 is an obtuse angle.

There is only one case that might cause trouble in the application of this formula. If the two lines are perpendicular,

(3) $$\alpha_1 = \alpha_2 \pm 90°.$$

Then, as we may recall from trigonometry,

(4) $$\tan\alpha_1 = -\cot\alpha_2 = -\frac{1}{\tan\alpha_2}$$

$$\text{or } m_1 = -\frac{1}{m_2} \text{ or } 1 + m_1 m_2 = 0.$$

In this case, $\tan\theta$ becomes formally a constant divided by zero, which we indicate by the symbol ∞, meaning that the tangent of θ is infinite. Conversely, when this happens, we note that the angle is 90°.

If one of the lines, say l_2, is vertical, so that $m_2 = \infty$, we divide both numerator and denominator of the fraction in (2) by m_2 and obtain the limit of the result as m_2 increases indefinitely.

EXAMPLE 1. Determine the angle between the lines

$$x - 2y = 4, \; x + 3y = 17.$$

By the methods of §III.6,

$$m_1 = \frac{1}{2}, \; m_2 = -\frac{1}{3}. \text{ Then } \tan\theta = \frac{\frac{1}{2} + \frac{1}{3}}{1 + \left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1.$$

Hence $\theta = 45°$.

EXAMPLE 2. Determine the angle that the line $3x + 2y = 2$ makes with the line $2x - 3y = 7$.

We have $m_1 = -\frac{3}{2}$, $m_2 = \frac{2}{3}$.

Since $m_1 m_2 + 1 = 0$ the lines are perpendicular to each other.

EXAMPLE 3. Find the angle that the line $2x + y = 5$ makes with the line $2x - 5 = 0$. (See Fig. 36.)

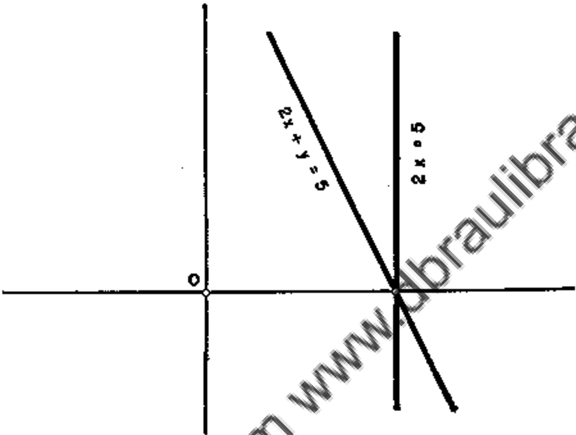

FIG. 36.—EXAMPLE 3.

The slope of the first line is -2, and the second line is perpendicular to the x-axis. Hence the angle which the first line makes with the second is

$$\theta = \text{arc tan } (-2) - \frac{\pi}{2}$$

$$= -\text{arc cot } (-2)$$

$$= \text{arc tan } \frac{1}{2}.$$

EXERCISES

1. Find the angle that the line $4x - y + 1 = 0$ makes with $2x + 3y - 5 = 0$. Plot both lines.
2. Find the angle between the lines $3x - 5y + 9 = 0$ and $10x + 6y - 7 = 0$.
3. Find the interior angles of triangle $A(2,-2)$, $B(8,4)$, $C(4,6)$.
4. Find the interior angles of the triangle whose vertices are $(-3,1)$, $(1,2)$, $(0,6)$.
5. Find the angle between the diagonals AC and BD of the quadrilateral $A(1,1)$, $B(4,-6)$, $C(4,3)$, $D(2,5)$.

Determine the angle between the pair of lines given by the following data and plot:

6. $2x + y = 5;\ x = 0.$
7. $2x + y - 5 = 0;\ x + 3 = 0.$
8. Through $(-1,-1)$ and $(1,5)$; through $(2,\frac{1}{2})$ and $(-3,-2)$.
9. Through $(7,\frac{1}{3})$ and $m = -2$; through $(0,0)$ and $(5,5)$.
10. $\frac{x}{5} + \frac{y}{8} = 1;\ y + 4 = 0.$
11. $\omega = \frac{\pi}{3},\ p = 4;\ \omega = \frac{2\pi}{3},\ p = 3.$
12. $\omega = \frac{\pi}{6},\ p = 6;\ \omega = \frac{2\pi}{3},\ p = 5.$
13. Find the equation of a line through $(4,13)$, making an angle of 45° with the line $x/(-4) + y/3 = 1$.
14. Given the line $2x - y = 4$. Find the equation of a second line through $(5,6)$ making an angle arc tan $(4/3)$ with the first. Draw the figure.
15. Show by means of (2) of III.10 that the angle between the lines $x \cos \omega_1 + y \sin \omega_1 - p_1 = 0$, $x \cos \omega_2 + y \sin \omega_2 - p_2 = 0$ is $\omega_1 - \omega_2$.

III.11 PARALLEL AND PERPENDICULAR LINES

Two lines are parallel if they have the same slope.

(1)
$$A_1x + B_1y + C_1 = 0,$$
$$A_2x + B_2y + C_2 = 0,$$

have the slopes $(-A_1/B_1)$, $(-A_2/B_2)$ which are equal provided that

(2)
$$A_1B_2 - A_2B_1 = 0 \quad \text{or} \quad \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0.$$

This is just another way of saying that the coefficients of x and y are proportional.

The two lines are perpendicular if the slope of one is the negative reciprocal of the slope of the other.

(3)
$$-\frac{A_1}{B_1} = \frac{B_2}{A_2}, \quad \text{or} \quad A_1A_2 + B_1B_2 = 0.$$

These relations hold except when the lines are perpendicular or parallel to the axes and the relations may be meaningless. But in this case the lines are obviously either parallel or perpendicular to each other.

It sometimes becomes necessary to write the equations of lines parallel or perpendicular to a given line. The easiest method of doing so is illustrated in the following.

EXAMPLE. Write the equations of the lines parallel and perpendicular, respectively, to the line $3x + 5y - 7 = 0$ and passing through the point (2,1). (See Fig. 37.)

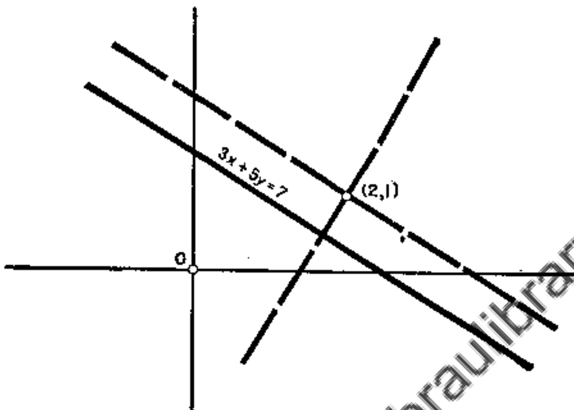

FIG. 37.

The slope of the given line is $-3/5$.
Hence by (5) of §III.8, the parallel line can be written

$$3x + 5y - (3 \cdot 2 + 5 \cdot 1) = 0,$$

or

$$3x + 5y - 11 = 0.$$

Since the slope of the perpendicular line is 5/3, we have

$$5x - 3y - (5 \cdot 2 - 3 \cdot 1) = 0,$$

or

$$5x - 3y - 7 = 0.$$

Note that for the perpendicular line, we can interchange the x- and y-coefficients, changing the sign of one of them and determining the constant term by the method discussed in §III.8.

EXERCISES

1. Are any of the following pairs of lines parallel, or perpendicular?
(a) $2x + 3y - 10 = 0$, $4x + 6y + 13 = 0$.
(b) $2x + 3y - 10 = 0$, $6x - 4y + 13 = 0$.
(c) $2x + 3y - 10 = 0$, $6x + 4y + 13 = 0$.
(d) $5x - y + 1 = 0$, $10x + 2y + 15 = 0$.
(e) $5x - y + 1 = 0$, $2x + 10y + 15 = 0$.
(f) $ax + by + c = 0$, $bx - ay + d = 0$.

2. Find the equations of the lines parallel and perpendicular, respectively, to the given line and passing through the given point:
(a) $5x - y + 7 = 0$, (1,4).
(b) $2x + 3y + 4 = 0$, (0,−5).
(c) $x + 12y - 4 = 0$, (−1,−2).
(d) $ax + by + c = 0$, (x_1,y_1).

3. Given a quadrilateral $A(-1,-3)$, $B(2,1)$, $C(9,2)$, $D(6,-2)$:
   (a) Write the equations of the four lines AB, BC, CD, DA. Show that they are parallel in pairs, and hence that the figure is a parallelogram.
   (b) Find the length of AD.
   (c) Find the distance of B from AD.
   (d) Find the area of the parallelogram in two ways.
   (e) Find the interior angle at A.
   (f) Write the equation of the line through B perpendicular to AD.
   (g) Show that the line of (f) meets AD in its mid-point.
4. Find the equations of the lines through the point $(0,9)$ perpendicular to the bisectors of the angles formed by the lines $x - y + 3 = 0$, $x + 7y - 5 = 0$.
5. (a) Plot the triangle whose vertices are $(2,-1)$, $(5,1)$, $(-1,10)$ and show by analytic geometry that the perpendicular bisectors of the sides are concurrent, and that the point of intersection is equidistant from the three vertices.
   (b) Repeat 5(a), using the triangle whose vertices are (x_1,y_1), (x_2,y_2), (x_3,y_3), thus proving the general theorem that the perpendicular bisectors of the sides of a triangle are concurrent and that the point of intersection is equidistant from the three vertices.
6. On a map 12 by 10 inches, a straight line runs from the N.E. corner to the mid-point of the opposite (West) side. A second line runs from four inches east of the S.W. corner, to a point three inches north of the same corner. By analytic geometry find the angle between the two lines and draw the bisector of the angle, which appears on the map.

SUMMARY OF CHAPTER III

The straight line in polar co-ordinates

$$\rho \cdot \cos(\theta - \omega) = p$$

The straight line in normal form.

$$x \cdot \cos\omega + y \cdot \sin\omega - p = 0$$

$$\frac{Ax + By + C}{\pm\sqrt{A^2 + B^2}} = 0$$

Distance of a point (x_1,y_1) from a line

$$x_1 \cos\omega + y_1 \sin\omega - p = d$$

$$\frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}} = d$$

Straight line through two points

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Point-slope form of equation of straight line

$$y - y_1 = m(x - x_1)$$

y-intercept-slope form

$$y = mx + b$$

Intercept form of equation of straight line

$$\frac{x}{a} + \frac{y}{b} = 1$$

Angle between two lines

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Parallel lines

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0$$

Perpendicular lines

$$A_1A_2 + B_1B_2 = 0$$

CHAPTER

IV

THE CIRCLE

IV.1 INTRODUCTION

We define a circle as the locus of points in a plane at a constant distance from a fixed point. In algebraic language, the condition, that the point (x,y) shall be at a fixed distance, r, from a fixed point (a,b), is, after radicals have been removed,

(1) $$(x - a)^2 + (y - b)^2 = r^2.$$

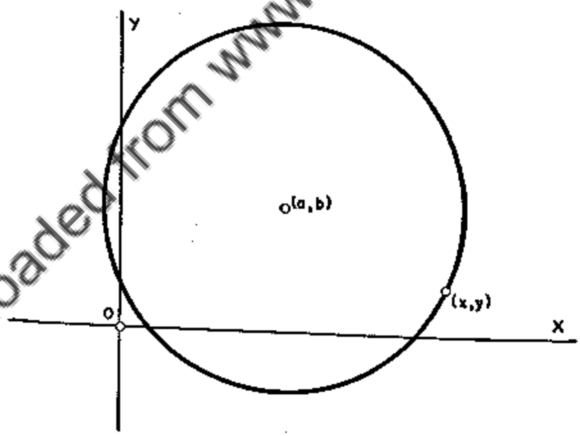

FIG. 38.—CIRCLE.

Any equation that can be reduced to this form represents a circle (Fig. 38), whose center and radius can be read at once.

EXAMPLE 1. Write the equation of the circle (Fig. 39) whose center is (2,1) and whose radius is 4.

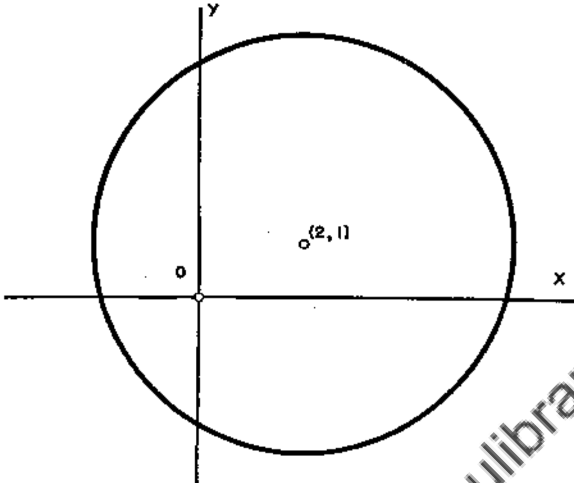

FIG. 39.—EXAMPLE 1. $(x-2)^2+(y-1)^2=16.$

EXAMPLE 2. What is the locus represented by the equation $x^2+y^2-6x+2y+1=0$?

Collecting the x-terms (and the y-terms) and completing squares, we have

$$(x^2-6x+9)+(y^2+2y+1)=9,$$

or

$$(x-3)^2+(y+1)^2=3^2.$$

Comparing with Equation (1), we see that this represents a circle with center at (3,−1) and radius 3 (Fig. 40).

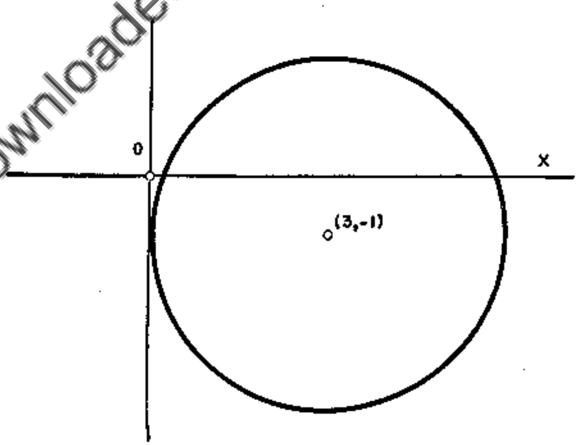

FIG. 40.—EXAMPLE 2.

A similar equation for the circle can be found in polar co-ordinates. We shall, however, limit the discussion to the two cases when the center is at the pole, or when the circle passes through the pole. Otherwise the form of the equation is too complicated for convenient use.

If the center is at the pole, the equation may be written at once and is

(2) $$\rho = r.$$

If the center lies on the polar axis, and the circle passes through the pole, the equation is

(3) $$\rho = 2r \cos \theta,$$

since, as is shown in Fig. 41, the triangle OAP, being inscribed

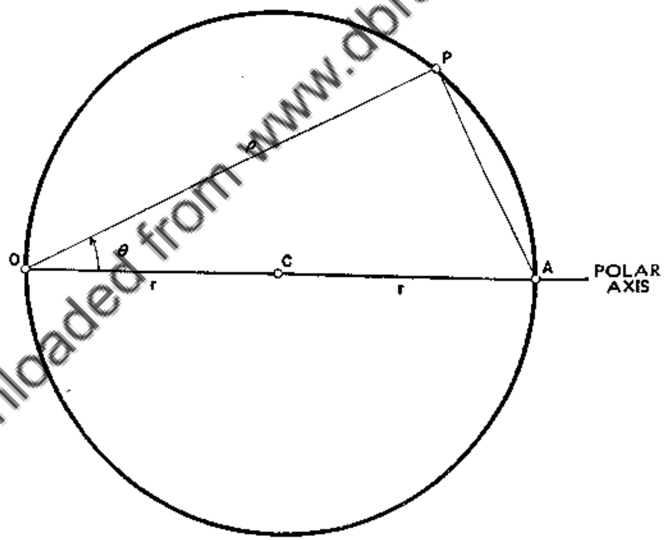

FIG. 41.—CIRCLE IN POLAR CO-ORDINATES.

in a semicircle, is a right triangle.

Similar reasoning shows that if the center is the point (r,α), the equation of the circle (Fig. 42) is

(4) $$\rho = 2r \cos(\theta - \alpha).$$

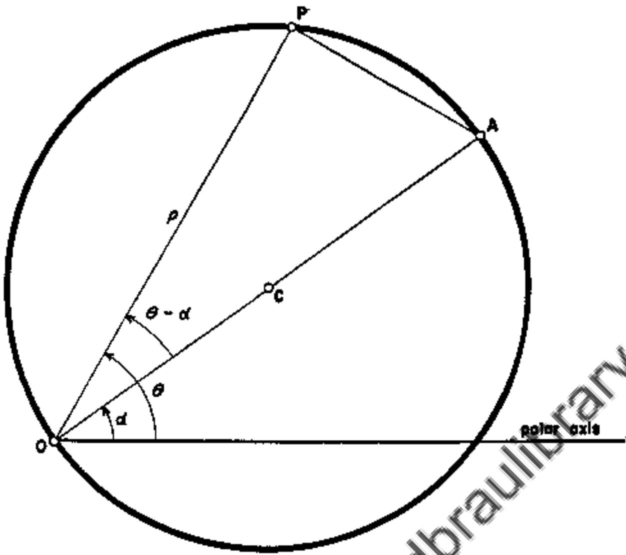

FIG. 42.—CIRCLE.

EXERCISES

1. In each of the following, determine the equation of the circle whose center is the point C and whose radius is r:

(a) $C(1,2)$, $r = 6$.
(b) $C(-5,0)$, $r = 2$.
(c) $C(0,-6)$, $r = \sqrt{3}$.
(d) $C\left(-\frac{5}{3}, -\frac{1}{2}\right)$, $r = 1$. (e) $C(-h,k)$, $r = 1$. (f) $C\left(\frac{4}{5},0\right)$, $r = 4$.

2. In each of the following circles, find the center and radius, and determine whether the given point lies on the circle:

(a) $(x-2)^2 + (y-1)^2 = 5$, $(4,0)$.
(b) $(x+2)^2 + y^2 = 6$, $(0,\sqrt{2})$.
(c) $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{2}{3}\right)^2 = 3$, $(0,0)$.
(d) $(x-1)^2 + (y+2)^2 = 17$, $(2,2)$.
(e) $(x+a)^2 + (y+b)^2 = 2(a^2+b^2)$, (b,a).

3. Given the center, C, and the radius, r, find the polar equation of the circle. Draw the graph.

(a) Center at pole, $r = 2$.
(b) $C\left(6,\frac{\pi}{3}\right)$, $r = 6$.
(c) $C(4,0)$, $r = 4$.
(d) $C\left(5,\frac{\pi}{2}\right)$, $r = 5$.
(e) $C\left(5,-\frac{\pi}{2}\right)$, $r = 5$. (g) $C\left(3,-\frac{\pi}{3}\right)$, $r = 3$.
(f) $C(4,\pi)$, $r = 4$.

4. Given the polar equation of the circle, find the center and radius. Determine whether the given point is on the curve. Draw the graph.

(a) $\rho = 5 \cos \theta$, $\left(\frac{5}{2}, \frac{\pi}{3}\right)$.

(b) $\rho = 6 \cos \left(\theta - \frac{\pi}{6}\right)$. $(-3\sqrt{3}, \pi)$.

(c) $\rho = 8 \sin \theta$, $\left(-8, \frac{\pi}{2}\right)$.

(d) $\rho = -16 \sin \theta$, $\left(-16, \frac{\pi}{2}\right)$.

(e) $\rho = 10 \cos \left(\theta + \frac{\pi}{3}\right)$. $\left(9, -\frac{\pi}{3}\right)$.

(f) $\rho = -4 \cos \theta$, $\left(2\sqrt{2}, \frac{\pi}{4}\right)$.

5. In the following, the center (in polar co-ordinates) and the radius are given. Find the equation of the circle in rectangular co-ordinates.

(a) $(6,0)$, $r = 6$. (b) $\left(4, -\frac{\pi}{3}\right)$. $r = 4$. (c) $\left(2, \frac{\pi}{6}\right)$. $r = 2$.

6. Find the equation of the circle with center at the origin and passing through the point $(-12,9)$.

7. Find the equation of the circle whose center is on the x-axis and the line $4x - 3y - 12 = 0$, and whose radius is the distance between the lines $4x - 3y - 12 = 0$ and $4x - 3y - 32 = 0$.

IV.2 GENERAL EQUATION OF A CIRCLE

The most general form of the equation of a circle is

(1) $$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0. \quad (A \neq 0.)$$

Since x^2 and y^2 have the same coefficient, we can divide each term by A, and then proceed, as in Example 2 of §IV.1, to reduce the equation to the standard form (1).

If the right hand member of the reduced equation is positive the locus is a circle, if zero, the locus is a point, and if negative there is no real locus and the equation defines an "imaginary circle."

EXAMPLE. Determine the locus whose equation is

$$3x^2 + 3y^2 - 12x + 4y = 0. \quad a \neq 0.$$

Dividing by 3, we have

$$x^2 + y^2 - 4x + \frac{4}{3}y = 0,$$

$$(x^2 - 4x + 4) + \left(y^2 + \frac{4}{3}y + \frac{4}{9}\right) = 4 + \frac{4}{9}.$$

The locus is a circle (Fig. 43) with center $(2, -\frac{2}{3})$ and radius $\frac{2}{3}\sqrt{10}$.

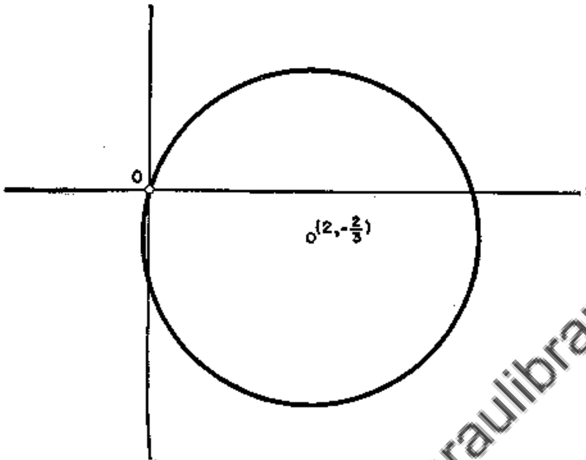

FIG. 43.

The student must be able to write the equation of any circle having a given center and radius, and to determine the center and radius of any circle whose equation is given.

EXERCISES

1. Find the center and radius of each of the following circles:
   (a) $x^2 + y^2 - 4x + 2y - 1 = 0.$
   (b) $x^2 + y^2 - \dfrac{2}{3}x + 4 = 0.$
   (c) $2x^2 + 2y^2 + 3x - 4y = 5.$
   (d) $9x^2 + 9y^2 + 12y + 4 = 0.$
   (e) $x^2 + y^2 + ax + by + c = 0.$
   (f) $x^2 + y^2 - 4x + 6y + 20 = 0.$ How many real points are on the circle (f)?
2. Reduce to rectangular co-ordinates and find the center and radius.
   (a) $\rho = 4 \cos \theta.$
   (b) $\rho = -12 \cos \theta.$
   (c) $\rho = \sin \theta.$
   (d) $\rho = 16 \cos \left(\theta - \dfrac{\pi}{3}\right).$ (e) $\rho = 16 \cos \left(\theta + \dfrac{\pi}{6}\right).$
3. Change to polar co-ordinates:
   (a) $x^2 + y^2 - 3x - 4y = 0.$ (b) $2x^2 + 2y^2 + 3x - y = 0.$
4. Show that the circles

$$x^2 + y^2 - 2x - 10y + 2 = 0$$

and

$$(x - 1)^2 + (y - 5)^2 = 25$$

are concentric.

5. Find the general equation of a circle with center at (5,8) and passing through the center of the circle $x^2 + y^2 - 2x - 10y + 1 = 0$.
6. Remembering that two circles are tangent externally or internally according as the distance between their centers equals the sum or difference of their radii, what can you say about the tangency of the following pairs of circles? Plot.
   (a) $x^2 + y^2 = 25, x^2 + y^2 - 18x + 65 = 0$.
   (b) $x^2 + y^2 - 4x - 6y + 9 = 0, x^2 + y^2 - 10x - 14y + 65 = 0$.
   (c) $x^2 + y^2 - 18x - 24y + 224 = 0, x^2 + y^2 - 16x - 16y + 64 = 0$.
7. Determine whether the point (2,4) is inside, on, or outside the circle $x^2 + y^2 = 25$.
8. (a) Determine the position of (12,8) relative to $x^2 + y^2 = 52$, without first plotting.
   (b) Find the equations of the circles having (12,8) for their center and tangent, internally and externally, respectively, to $x^2 + y^2 = 52$.
9. (a) What are the x-intercepts of $x^2 + y^2 - 24x - 12y + 80 = 0$? At these intercept points, find the equations of the tangents to the circle. Draw the graph.
   (b) By analytic geometry, show that the bisector of the angle between the tangents in 9(a) passes through the center of the circle.
10. Given the circle $x^2 + y^2 = 25$, and the chord $P_1(0,5)P_2(4,3)$. Show by analytic geometry that the perpendicular bisector of P_1P_2 passes through the center.
11. Repeat Exercise 10 for the circle $x^2 + y^2 = r^2$ and the chord $P_1(x_1,y_1)P_2(x_2,y_2)$.
12. By analytic geometry, prove that the perpendicular from the center of $x^2 + y^2 = r^2$ to a chord $P_1(x_1,y_1)P_2(x_2,y_2)$ passes through the midpoint of P_1P_2.
13. Given the circle $x^2 + y^2 = a^2$ and a point $P_1(x_1,y_1)$ on this circle and in the first quadrant. Draw the perpendicular P_1C to the diameter $A(-a,0)B(a,0)$, where C is a point on the x-axis, and prove that P_1C is the mean proportional between AC and CB (i.e. $\overline{P_1C}^2 = \overline{AC}\cdot\overline{CB}$).

IV.3 CIRCLE THROUGH THREE POINTS

We recall from plane geometry that the center of a circle through three points, not on a straight line, is the point of intersection of the perpendicular bisectors of any two of the line segments determined by the three points. Thus, if we find the equations of two of these bisectors, solve them simultaneously, and use the resulting point of intersection as center, with a radius equal to the distance of this point from one of the given points, the equation of the circle through the three points can be determined.

This is a special case of the general problem of finding the equation of a circle when any three conditions which fix the circle are given.

EXAMPLE. Determine the equation of the circle (Fig. 44) through the three points (4,3), (2,1), (5.2,−.6).

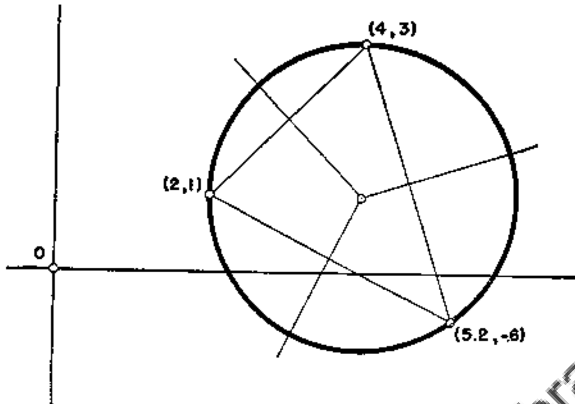

FIG. 44.—CIRCLE THROUGH THREE POINTS.

Two of these perpendicular bisectors are

$$x + y - 5 = 0,$$
$$x - 3y - 1 = 0,$$

and they intersect in (4,1) which is at a distance $r = 2$ from each of the given points. The required equation of the circle is

$$(x - 4)^2 + (y - 1)^2 = 4,$$

or $$x^2 + y^2 - 8x - 2y + 13 = 0.$$

EXERCISES

1. Work out the solution of the foregoing illustrative example.
2. Determine the equation of the circle through (0,5), (3,4), and (0,1).
3. Write the equation of the circle through (2,−1), (−4,5), and (0,5).
4. Find the circle determined by (6,1), (−2,−3), and (10,7).
5. The equations of the sides of a triangle are
$$x + 2y - 9 = 0,$$
$$2x - y - 3 = 0,$$
$$x - 2y + 15 = 0.$$
Find the equation of the circumscribed circle.
6. Plot the quadrilateral with vertices at (1,3), (5,0), (8,4), and (4,7). Determine whether or not these points are all on the same circle.

IV.4 INTERSECTION OF A STRAIGHT LINE AND A CIRCLE

The problem of finding the intersections of a straight line and a circle is really a problem of algebra, involving the solution of a pair of equations, one linear and one quadratic. The method is illustrated in the following example:

EXAMPLE. Find the common points of the straight line $x + 3y - 7 = 0$ and the circle $7x^2 + 7y^2 + 11x - 23y = 0$, shown in Fig. 45.

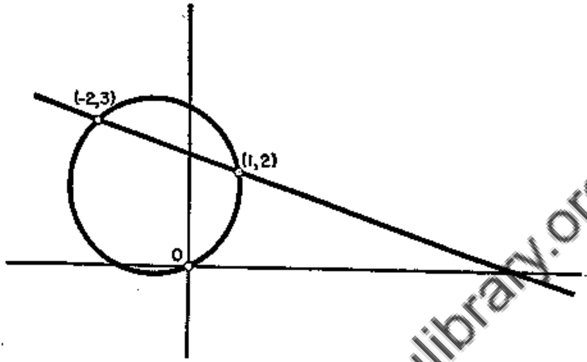

FIG. 45.

From the first equation, we have

$$x = 7 - 3y.$$

Substituting in the second,

$$343 - 294y + 63y^2 + 7y^2 + 77 - 33y - 23y = 0.$$

After simplifying by collecting terms and dividing by a common numerical factor, we have the equation

$$y^2 - 5y + 6 = 0.$$

From this we obtain $y = 2$ or 3, and then, by substitution, $x = 1$ or -2. Hence, the desired points are $(1,2)$, $(-2,3)$.

There are three possible cases of the intersection of a straight line and a circle, corresponding to the three possibilities of the solution of a quadratic equation in one unknown.

When we eliminate one variable from the equations of a straight line and a circle, we have a quadratic equation in the remaining variable, of the general form

$$ax^2 + bx + c = 0.$$

This equation has either real and distinct roots, real and equal roots, or imaginary roots, according as the discriminant,

$$b^2 - 4ac,$$

is positive, zero, or negative. Corresponding to these three cases, a straight line meets a circle at two real and distinct points, is tangent to the circle, or meets it at an imaginary point (no real intersection), according to the values of the discriminant of the quadratic equation found by eliminating one variable.

EXERCISES

In Exercises 1–7, find the points of intersection of the given line and circle, and state whether these points are real and distinct, real and coincident, or imaginary.

1. $y = x + 1$, $x^2 + y^2 = 25$.
2. $y = x + 5$, $x^2 + y^2 = 25$.
3. $y = 0$, $x^2 + y^2 - 8x + 6y + 16 = 0$.
4. $x - y = 0$, $x^2 + y^2 - 12y + 27 = 0$.
5. $x = 0$, $x^2 + y^2 + 8x - 12y + 48 = 0$.
6. $2x + y = 3$, $x^2 + y^2 - 3x + 1 = 0$.
7. $2x + y\sqrt{3} - 17 = 0$, $x^2 + y^2 - 8x + 7 = 0$.
8. For what values of k will the points of intersection of $2x + y = k$ and $x^2 + y^2 = 5$ coincide? (Eliminate one variable and set the discriminant of the resulting quadratic equal to zero.)
9. For what value of m will the points of intersection of $mx - y = m - 2$ and $x^2 + y^2 = 5$ coincide?
10. Find the equation of the tangent from the point (0,8) to the circle $x^2 + y^2 - 4x - 4y = 0$. (Write the equation of a line through (0,8) with slope m. The points of intersection of this line with the circle must coincide.)
11. Find the equations of the tangents from (0,1) to the circle with center (3,2) and radius $\sqrt{2}$.

SUMMARY OF CHAPTER IV

The circle with center (a,b) and radius r

$$(x - a)^2 + (y - b)^2 = r^2,$$

or

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

In polar co-ordinates

$$\rho = r$$
$$\rho = 2r \cos \theta$$
$$\rho = 2r \cos (\theta - \alpha)$$

General equation of a circle

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0.$$

$$\text{Center: } \left(-\frac{G}{A}, -\frac{F}{A}\right)$$

$$\text{Radius: } \frac{1}{A}\sqrt{G^2 + F^2 - CA}$$

Straight line tangent to a circle
substitute $y = mx + b$ in general equation

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0$$

Set discriminant of equation equal to zero and solve for m or b

REVIEW OF CHAPTERS II, III, IV

1. (a) Are the three points whose polar co-ordinates are $(8\sqrt{2}, \pi/2)$, $(8,\pi/4)$, $(8\sqrt{2},0)$ collinear? Test by using (1) distances, (2) areas.
   (b) Change the polar co-ordinates of the three points of 1(a) to rectangular co-ordinates and test for collinearity.
2. Find the polar equation of the line $x + y - 8\sqrt{2} = 0$.
3. Write the equation of the line determined by the points, $(-3,4)$, $(6,13)$, and reduce that equation to the point-slope, the general, the normal, and the polar forms.
4. Draw the lines given by the following data:
   (a) Point $(-3,4)$, slope $= -\frac{5}{3}$.
   (b) $\omega = 120°$, $p = 4$.
   (c) $2x + 4y - 5 = 0$.
   (d) $\rho \cos\left(\theta + \frac{\pi}{6}\right) = 3$.
5. Find the co-ordinates of a point dividing the segment $P_1(-6,0)$ $P_2(0,2)$ in the ratio $-5/2$, and check the result.
6. Draw the following lines and find the angle which the first line makes with the second.
   (a) $3x - 7y + 4 = 0$, $6x - 14y + 5 = 0$.
   (b) $x + 3y - 9 = 0$, $24x - 8y + 13 = 0$.
   (c) $4x + 2y + 15 = 0$, $12x - 5y + 30 = 0$.
7. (a) Plot the points $A(3,8)$, $B(-2,-4)$, $C(10,1)$. Find the co-ordinates of a fourth point, D, such that $ABCD$ is a rhombus.
   (b) Show that the diagonals of the figure $ABCD$ are perpendicular.
   (c) By analytic geometry, show that BD bisects the angles B and D.
8. Find the length of the altitude, drawn through (x_1,y_1), of the triangle (x_1,y_1), (x_2,y_2), (x_3,y_3).
9. Write both the polar and rectangular equations of the circle
   (a) whose center in rectangular co-ordinates is $(5,0)$, and whose radius is 5.
   (b) with radius 4, and center, in polar co-ordinates, $(4,\pi/6)$.
10. Reduce $x^2 + y^2 - 6x - 6\sqrt{3}y = 0$ to polar form, and find the center and radius of the circle.
11. (a) Find the equation of the circle through $A(-5,0)$, $B(3,4)$, $C(\sqrt{5},2\sqrt{5})$.
    (b) Through the point $D(0,5)$, a chord DE is drawn parallel to AB. Find the co-ordinates of E.
12. Given the circle, $x^2 + y^2 = 25$, find the equation of a circle through $(6,17)$ and externally tangent to the given circle at the point $(4,3)$.
13. Find the equations of the tangents through $(4,7)$ to the circle $x^2 + y^2 = 13$. Find the points of tangency.
14. Find the common points of the circles $x^2 + y^2 - 2x + 8y - 8 = 0$ and $2x^2 + 2y^2 + 5y = 0$.
15. Find the equation of the circle inscribed in the triangle whose sides are the lines

$$x - 2y + 15 = 0,$$
$$2x - y = 3,$$
$$x + 2y - 9 = 0.$$

CHAPTER

V

CONIC SECTIONS

V.1 DEFINITION OF A CONIC

The ancient Greeks studied the curves known as conic sections from the standpoint of actual sections of a cone by a plane. We **could** do the same thing, but find it much more convenient to take up their properties from a different point of view.

A **conic section** (Fig. 46) is defined as the locus of a point such that the ratio of the distance of this point from a fixed point (the **focus**), to the distance of the point from a fixed line (the **directrix**), is a constant (the **eccentricity**).

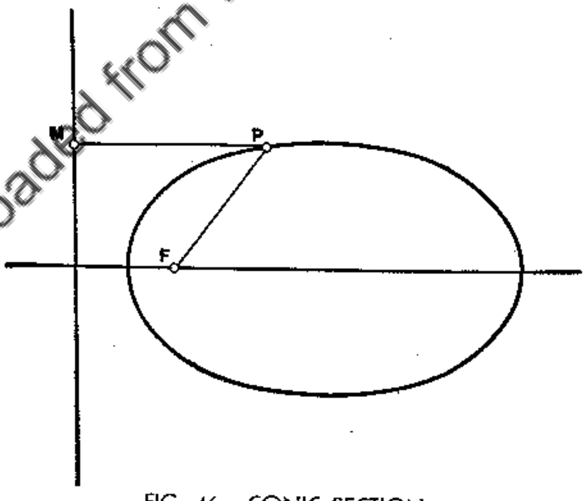

FIG. 46.—CONIC SECTION.

These curves have been found of great importance, first because they occur very frequently in nature, secondly, and this is in part a consequence of the first, because of very important applications. We shall, of course, in this text not consider their appeal to the esthetic sense with their beauty of form and symmetry, nor discuss the very interesting fact that the heavenly bodies in the solar system—the planets and the occasional comets—move in orbits that are conic sections.

The observant student will recall having seen bridge arches in the shape of ellipses, or parabolas, which combine beauty of form with great strength.

V.2 EQUATION OF A CONIC—GENERAL FORM

Let us assume the focus of a conic at (m,n), and that the directrix is the line (Fig. 47)

(1) $$ax + by + c = 0.$$

Let the eccentricity be e. Then, from the definition,

(2) $$\frac{\sqrt{(x-m)^2 + (y-n)^2}}{\left|\dfrac{ax + by + c}{\sqrt{a^2 + b^2}}\right|} = e.$$

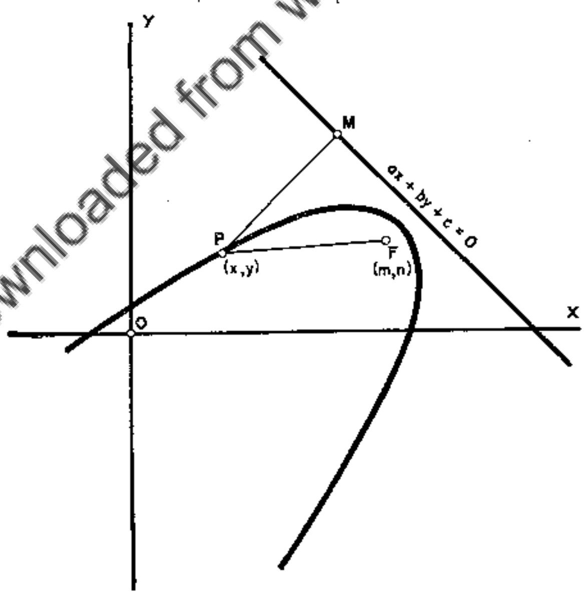

FIG. 47.—CONIC SECTION.

Clearing of fractions and radicals, and collecting terms, we have an equation of the form

(3) $$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

where A,B,C,H,G,F are functions of a,b,c,m,n,e. As a result, we state that

The equation of a conic section is of the second degree in x and y.

The converse problem, of finding the locus of the general equation of the second degree in x and y, is much more complex, except in special cases, and is handled, when necessary, by more or less indirect methods. Various features of the problem will be considered later and means devised for determining the foci, directrices, and eccentricity when the locus is a real conic.

EXERCISES

In each of the following, write the equation of the conic with the given focus, directrix, and eccentricity.

1. $F(4,2)$, $x + y - 1 = 0$, $e = 1$.
2. $F(-3,1)$, $x - 2y = 3$, $e = 2$.
3. $F(2,0)$, $x - 11 = 0$, $e = \frac{1}{2}$.
4. $F(-2,-4)$, $x + 5 = 0$, $e = 1$.
5. $F(-4,8)$, $3y - 49 = 0$, $e = \frac{3}{5}$.
6. $F(1,2)$, $5y + 6 = 0$, $e = \frac{5}{3}$.
7. $F(1,0)$, $x + 1 = 0$, $e = 1$.
8. $F(0,3\sqrt{3})$, $y - 4\sqrt{3} = 0$, $e = \frac{\sqrt{3}}{2}$.
9. $F(10,0)$, $5x - 32 = 0$, $e = \frac{5}{4}$.
10. In the development of §V.2, show that if $e = 1$, Equation (3) starts out as
$$b^2x^2 - 2abxy + a^2y^2 + \dots = 0,$$
that is, the second-degree terms form a perfect square.
11. (a) Show that in Equation (3) of §V.2:
$$\begin{aligned} A &= (1 - e^2)a^2 + b^2, \\ B &= a^2 + (1 - e^2)b^2, \\ H &= -abe^2, \\ G &= -m(a^2 + b^2) - ace^2, \\ F &= -n(a^2 + b^2) - bce^2, \\ C &= (a^2 + b^2)(m^2 + n^2) - c^2e^2. \end{aligned}$$
(b) Write the equation of a conic for which $e = 0$. What is the locus?

V.3 EQUATION OF A CONIC—POLAR FORM

As we have seen before, the sensible thing, when studying any particular problem, is to choose the easiest way. For example, in studying the properties of a conic section, in general, we should choose the position of the co-ordinate system in such a way as to make the algebra as simple as possible.

In polar co-ordinates, the distance of a point from the pole is the radius vector, ρ. The simplest equation of a straight line not passing through the pole is

$$(1) \qquad \rho = p \sec \theta,$$

where p is the perpendicular distance from the pole to the line (Fig. 48).

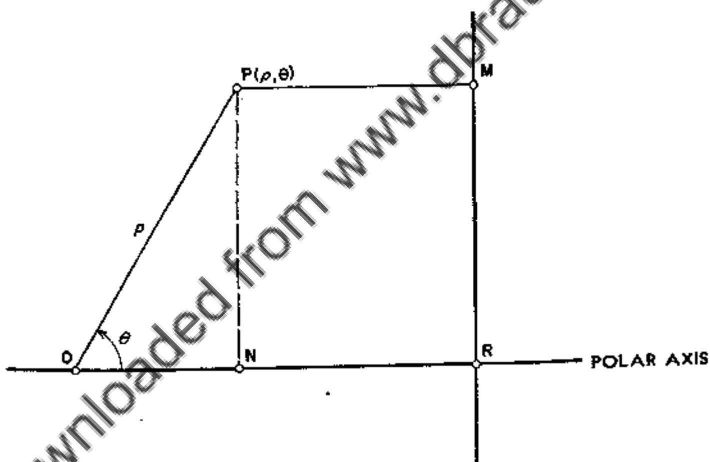

FIG. 48.—POLAR EQUATION OF CONIC.

Then, if we assume the pole as focus, and the line (1) as directrix, it follows that

$$\frac{OP}{PM} = e, \; OR = p, \; OP = \rho,$$

$$ON = \rho \cos \theta,$$

$$PM = NR = OR - ON = p - \rho \cos \theta.$$

The equation of the conic is seen to be

(2) $$\frac{\rho}{p - \rho \cos \theta} = e.$$

On clearing of fractions and solving for ρ, we have

(3) $$\rho = \frac{ep}{1 + e \cos \theta}.$$

If we had chosen the directrix to the left of the focus, with the equation

(4) $$\rho = -p \sec \theta,$$

the equation of the conic would have been

(5) $$\rho = \frac{ep}{1 - e \cos \theta}.$$

EXERCISES

1. Show how Equation (3) is derived from Equation (2) in §V.3.
2. Make a drawing of the case in which the directrix is to the left of the pole, and show how Equation (5) is developed from the figure.
3. Write the polar equations of the conics for the following data:
   (a) $e = 1, p = 10$, directrix to the right.
   (b) $e = 1, p = 6$, directrix to the left.
   (c) $e = 3, p = 15$, directrix to the right.
   (d) $e = \frac{2}{3}, p = 4$, directrix to the right.
   (e) $e = 2, p = 12$, directrix to the left.
   (f) $e = \frac{1}{4}, p = 3$, directrix to the right.
4. Draw the conics of 3(a), (b), (e), (f).
5. Find the polar equation of the conic with $e = 1$, if the focus is at the pole, and the directrix is perpendicular to the polar axis and passes through the point $(4, \pi/3)$.
6. Find the equation of the conic for which $e = 2$, the focus is at the pole, and the directrix is the line $\rho \cos \theta = 5$.
7. Reduce the following polar equations of conics to rectangular coordinates:
   (a) $\rho = \frac{2}{1 + \cos \theta}$.
   (b) $\rho = \frac{8}{1 + 2 \cos \theta}$.
   (c) $\rho = \frac{6}{1 - 2 \cos \theta}$.
   (d) $\rho = \frac{10}{2 - \cos \theta}$.
8. Construct a figure and from it derive equation (5) of V.3.
9. Obtain the equation of the conic with $\rho = p \csc \theta$ as directrix.

V.4 THE PARABOLA IN RECTANGULAR CO-ORDINATES

The parabola is a conic whose eccentricity is $e = 1$. If the focus is the point $(a,0)$, and the directrix is the line

(1) $$x + a = 0,$$

the equation is

(2) $$\frac{\sqrt{(x-a)^2 + y^2}}{|x+a|} = 1. \text{ (See Fig. 49.)}$$

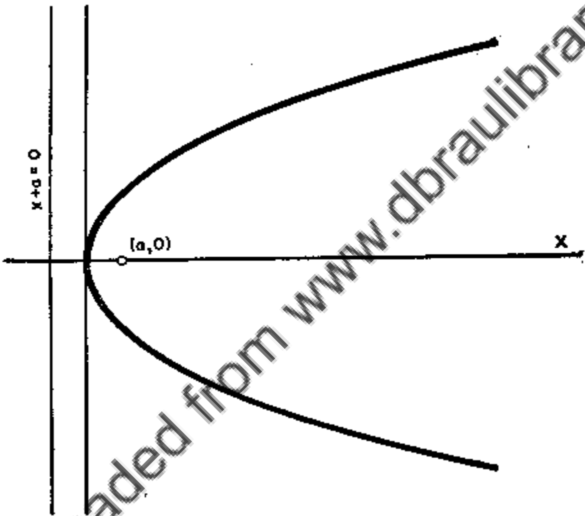

FIG. 49.—PARABOLA $y^2 = 4ax$.

If we clear of fractions and radicals, we have

(3) $$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2,$$

which reduces at once to

(4) $$y^2 = 4ax.$$

The straight line through the focus and perpendicular to the directrix is an axis of symmetry for any conic, and is called the **principal axis.** That this axis is an axis of symmetry in the present curve is shown by the fact that for every value of x there are two values of y, numerically equal but opposite in sign.

If a is positive there are no real values of y corresponding to negative values of x. Hence, the curve lies entirely to the right of the y-axis, and for all positive values of x there correspond real values of y. Hence, the curve is unlimited in extent in the direction of the positive x-axis.

Obviously, if the value of a were negative, the direction of the curve would be reversed. If the co-ordinates x and y were interchanged, the y-axis would be the axis of symmetry or principal axis of the curve, which would extend up or down according to the algebraic sign of a.

The point where a conic crosses its principal axis is known as the **vertex.**

The chord passing through the focus and perpendicular to the axis is called the **latus rectum,** and its length is $4a$, as can be verified from the equation by setting $x = a$.

EXAMPLE 1. Write the equation of the parabola with focus at $(-1,0)$ and directrix $x = 1$. Draw the curve. It follows from §V.4 that the desired equation is

$$y^2 = -4x.$$

The curve is symmetrical with respect to the x-axis, has its vertex at the origin, and lies entirely to the left of the y-axis (Fig. 50). The ends

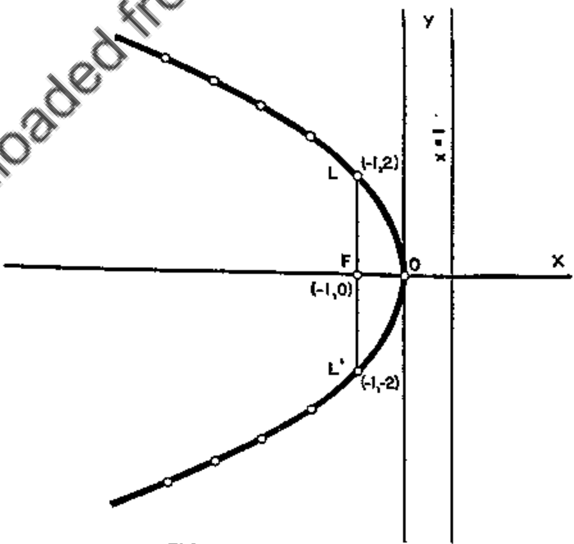

FIG. 50.—EXAMPLE 1.

of the *latus rectum* are $(-1,2)$, $(-1,-2)$. The points at which the curve cuts the line $x = -2$ are at a distance 3 from F (in a parabola, distance from focus = distance from directrix) and can be located geometrically without computation. Additional points may be located in a similar manner.

EXAMPLE 2. Draw the curve $x^2 = 2y$. Determine the focus, directrix, and ends of the *latus rectum*.

The curve is a parabola, with the y-axis as principal axis (Fig. 51).

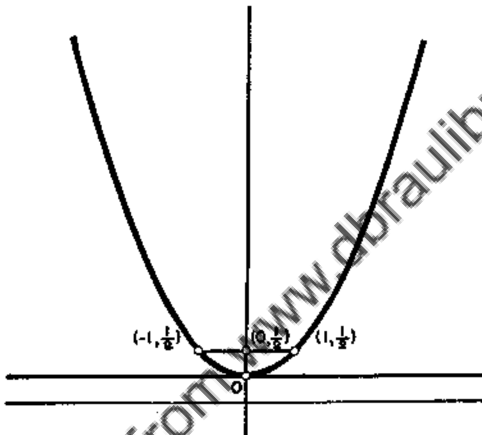

FIG. 51.—EXAMPLE 2.

Since $4a = +2$, the curve extends upward. The focus is at a distance $a = \frac{1}{2}$ up from the vertex, $(0,\frac{1}{2})$. The ends of the *latus rectum* are $(1,\frac{1}{2})$, $(-1,\frac{1}{2})$. The directrix is

$$y = -\frac{1}{2}.$$

Additional points can be found as in Example 1.

EXERCISES

Find the equation of the conic for which

1. $e = 1$, $F = (-a,0)$, directrix is $x - a = 0$.
2. $e = 1$, $F = (0,a)$, directrix is $y + a = 0$.
3. $e = 1$, $F = (0,-a)$, directrix is $y - a = 0$.

In each of the following, plot the parabola, the focus, *latus rectum*, directrix, and find the co-ordinates of the focus and of the ends of the *latus rectum*, and the equation of the directrix:

4. $y^2 = 6x$.
5. $y^2 = -8x$.
6. $y^2 - x = 0$.
7. $x^2 = 6y$.
8. $\frac{1}{4}x^2 + y = 0$.
9. $x^2 + 8y = 0$.
10. $x^2 = y$.

Plot the parabola with vertex at the origin and with

11. Focus at (5,0).
12. Length of *latus rectum*, 16. Equation of *latus rectum*, $y = 2$.
13. Equation of *latus rectum*, $y = 2$.
14. Equation of directrix, $y + 2 = 0$.
15. (a) Show that the line $x - 2y + 4 = 0$ is tangent to the parabola $y^2 = 4x$, and find the point of tangency.
    (b) A line joining the focus to a point on a conic is called the focal radius to the point in question. Show that the tangent in 15(a) bisects the angle between the focal radius to the point of tangency and the parallel to the axis of the curve.
16. Find the points of intersection of the circle $x^2 + y^2 = 10$ and the parabola $y^2 = 3x$.
17. Find the points of intersection of the parabolas $y^2 = 8x$ and $x^2 = -2\sqrt{2}y$.

V.5 EQUATION OF A CENTRAL CONIC

We have seen that when $e = 1$ the conic is a parabola. When $e \neq 1$ it is called a **central conic** for reasons that we shall see presently.

Let us write the equation of a conic section with the focus at $(ae,0)$ and the directrix $ex - a = 0$, where $e \neq 1$, as in Fig. 52:

(1) $$\frac{\sqrt{(x-ae)^2 + y^2}}{\left|x - \frac{a}{e}\right|} = e.$$

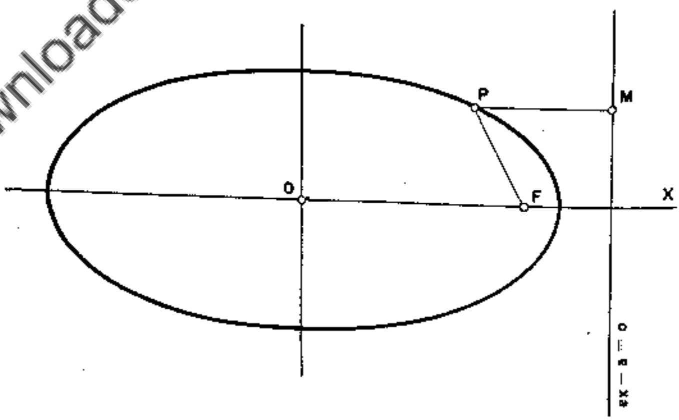

FIG. 52.—ELLIPSE.

If we clear of fractions and square, the equation becomes,

(2) $$x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2.$$

This equation may be reduced to the form

(3) $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

If the focus had been taken at $(-ae, 0)$ and the directrix as the line $ex + a = 0$, the equation of the conic would have been identical with equation (3). Therefore every central conic has two foci and two directrices. The proof of this is left as an exercise for the student.

If we write

(4) $b^2 = a^2(1 - e^2)$ for e less than one,
$b^2 = a^2(e^2 - 1)$ for e greater than one,

the equation of the conic becomes

(5a) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ for } e < 1,$$

(5b) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ for } e > 1.$$

If the eccentricity is less than one, the curve is an **ellipse**. This is a closed curve, of limited extent, since there are no real points of the curve for which x is greater than a in absolute value, or for which y is greater than b in absolute value.

If the eccentricity exceeds 1, the curve is a **hyperbola.** This curve has real points only if x is greater than or equal to a in absolute value, and is unlimited in extent.

The x-axis, in the preceding, is the **principal axis** of the curve. The y-axis, also an axis of symmetry, is called the **secondary axis** (minor axis for the ellipse, conjugate axis for the hyperbola). If the equation of the directrix were $ey - a = 0$, and the focus were at $(0,ae)$, the principal axis would be vertical, and the a^2 would be the denominator of y^2.

EXAMPLE 1. The curve $x^2/9 + y^2/16 = 1$ is an ellipse with principal axis vertical, since the larger denominator, 16, which is a^2 for the ellipse, goes with the y^2.

EXAMPLE 2. The curve $x^2/9 - y^2/16 = 1$ is a hyperbola with principal axis horizontal. (x^2-term positive.)

EXERCISES

1. Show the reduction from Equation (1) to Equation (3).
2. Using the focus $(0,ae)$ and directrix $ey - a = 0$, derive the equation $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.
3. In the following sight exercises, name the conic and the direction of the principal axis:

(a) $\frac{x^2}{4} + \frac{y^2}{9} = 1.$

(b) $\frac{x^2}{10} + \frac{y^2}{-12} = 1.$

(c) $\frac{-x^2}{16} + \frac{y^2}{4} = 1.$

(d) $\frac{x^2}{4} + y^2 = 1.$

(e) $x^2 - \frac{y^2}{4} = 1.$

(f) $16x^2 + 9y^2 = 144.$

(g) $4x^2 - 9y^2 = -36.$

(h) $25y^2 + x^2 = 225.$

(i) $16x^2 - 8y^2 = -4.$

(j) $2x^2 + y^2 = 30.$

(k) $3x^2 - 2y^2 = 12.$

(l) $2x^2 + 5y^2 = 5.$

4. By taking the focus at $(-ae,0)$ and directrix $ex + a = 0$, show that since the resulting equation is the same as Equation (3), every conic with eccentricity not equal to 1 has two foci and two directrices.
5. (a) If $b^2 = a^2(1 - e^2)$, find e in terms of a and b.
   (b) If $b^2 = a^2(e^2 - 1)$, find e in terms of a and b.
   (c) Show that the length of the *latus rectum* is $\frac{2b^2}{a}$.
6. Reduce the following to the standard form (5), and find e, the foci, directrices, and length of the *latus rectum*:

(a) $25x^2 + 9y^2 = 225.$

(b) $9x^2 - 16y^2 = 144.$

(c) $3x^2 - y^2 = -9.$

(d) $4x^2 + y^2 = 1.$

(e) $-x^2 + y^2 = 9.$

(f) $16x^2 - y^2 = -25.$

V.6 FORMS OF THE CENTRAL CONICS

The ellipse and the hyperbola are called central conics, since each has a point (the origin in the cases given in the preceding section) such that any line through that point meets the curve in two points equally distant from this particular point. Such a point is called a **center of symmetry.**

Consider the equation of the ellipse,

(1) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This curve cuts the axes in the points $(\pm a,0)$, $(0,\pm b)$. Solving for y gives

(2) $$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$

From this we see that y is real only if x^2 is less than a^2. Thus x is restricted to values between $-a$ and $+a$. Similarly we can show that y is restricted to values between $-b$ and $+b$. Hence the ellipse is a finite closed curve, as in Fig. 53.

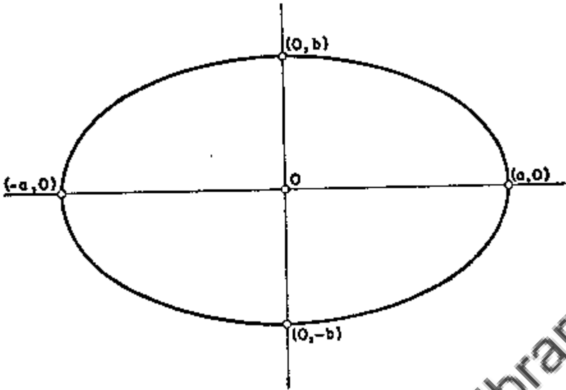

FIG. 53.—CENTRAL CONIC.

The standard equation of the hyperbola differs from the equation of the ellipse in the sign of b^2. The equation of the hyperbola is

(3) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Hence

(4) $$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$$

which shows that x cannot be less than a in absolute value, but that y is real for all other values of x, no matter how large. The hyperbola consists of two branches, as shown in Fig. 54.

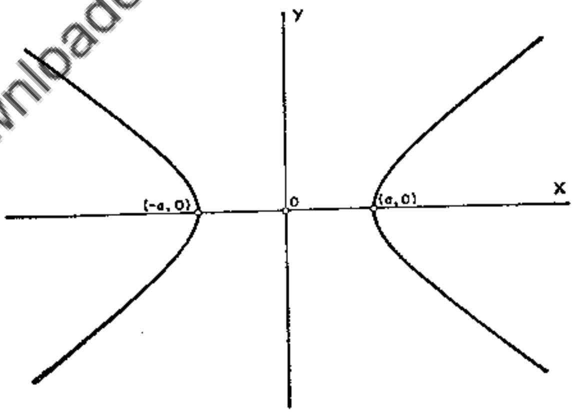

FIG. 54.—CENTRAL CONIC.

EXAMPLE 1. Sketch the graph of $16x^2 + 9y^2 = 144$.

Dividing through by 144, we have

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

The curve is an ellipse (Fig. 55) with principal axis vertical ($a^2 = 16$

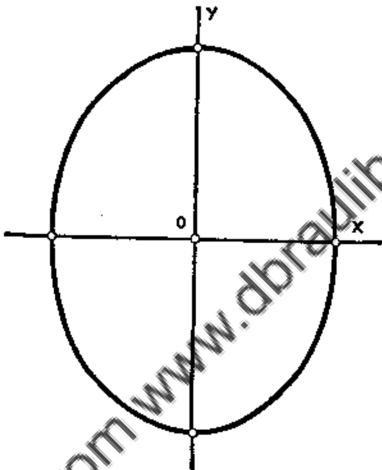

FIG. 55.—EXAMPLE 1.

under y^2 and both terms positive). The foci are at a distance $ae = \sqrt{a^2 - b^2} = \sqrt{7}$ from the center, which lies at the origin. They are $(0,\sqrt{7})$, $(0,-\sqrt{7})$. The vertices are at $(0,4)$, $(0,-4)$. The curve crosses the x-axis at $(3,0)$, $(-3,0)$. The ends of the *latera recta* are $(9/4,\sqrt{7})$, $(-9/4,\sqrt{7})$, $(9/4,-\sqrt{7})$, $(-9/4,-\sqrt{7})$. The directrices are $y = \pm a/e = \pm 16/\sqrt{7}$.

EXAMPLE 2. Sketch the graph of $9x^2 - 4y^2 + 36 = 0$.

Reduce to standard form (transpose 36 and divide by -36).

The result is

$$\frac{x^2}{-4} + \frac{y^2}{9} = 1.$$

Since one term is negative, the curve is a hyperbola, and since the positive term is the y^2-term, the principal axis is vertical. There is no point with an ordinate less than 3 in absolute value. For every value of x there are two real values of y (Fig. 56).

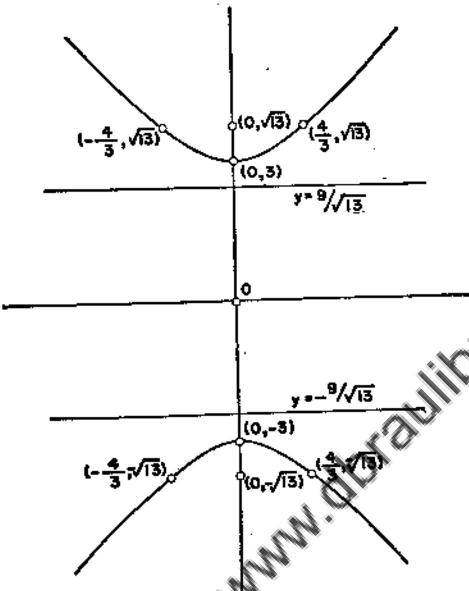

FIG. 56.—EXAMPLE 2.

$ae = \sqrt{a^2 + b^2} = \sqrt{13}$. The foci are $(0, \pm\sqrt{13})$. The ends of the *latera recta* are $(\pm 4/3, \pm\sqrt{13})$. The directrices are $y = \pm 9/\sqrt{13}$.

EXERCISES

1. Sketch the following conics, showing foci, *latera recta*, and directrices:

(a) $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1.$

(b) $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1.$

(c) $9x^2 - 4y^2 = 36.$

(d) $x^2 + 8y^2 = 8.$

(e) $4x^2 - y^2 = -4.$

(f) $x^2 - y^2 = 4.$

(g) $-x^2 + 9y^2 = 9.$

(h) $x^2 - 3y^2 + 9 = 0.$

2. Find the equation of the conic for the given conditions:

(a) $e = \dfrac{1}{2}$, $F_1 = (3,0)$, $F_2 = (-3,0)$.

(b) $e = \dfrac{4}{3}$, $F_1 = (0,4)$, $F_2 = (0,-4)$.

(c) $e = \dfrac{2}{5}$, center $(0,0)$, principal diameter $= 20$ along y-axis.

(d) $e = \dfrac{5}{2}$, directrices $x - 4 = 0$, $x + 4 = 0$.

3. Given the ellipse $x^2/25 + y^2/16 = 1$, show that the line joining the end of the *latus rectum* to the intersection of the principal axis with the nearest directrix, is tangent to the curve. In what way might this property help in sketching the curve? (Hint: Solve the equations of the line and the ellipse for their points of intersection, and show that they coincide.)
4. Given the hyperbola $x^2/16 - y^2/9 = 1$, show that the line joining the end of a *latus rectum* to the intersection of the principal axis with the nearest directrix, is tangent to the curve.
5. By using the equation of the general conic, $x^2/a^2 \pm y^2/b^2 = 1$, prove that the property described in Exercises 3 and 4 is general.
6. Find the points of intersection of the following pairs of conics:

   (a) $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1,\ y^2 = \dfrac{9x}{4}.$

   (b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1,\ \dfrac{y^2}{4} - \dfrac{x^2}{16} = 1.$

   (c) $x^2 + y^2 = 13,\ \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1.$
7. Show that the sum of the focal radii drawn to the point (3,12/5) on the ellipse $x^2/25 + y^2/9 = 1$ equals the length of the principal diameter.
8. Show that the locus of a point such that the sum of its distances from two fixed points is a constant greater than the distance between the fixed points is an ellipse with these two points as foci. (Hint: Let the two points be $(ae,0)$, $(-ae,0)$ and the constant $= 2a$.)
9. Show that the locus of a point the difference of whose distances from two fixed points is a constant less than the distance between the fixed points, is a hyperbola with these two points as foci.

SUMMARY OF CHAPTER V

Equation of a conic section

Rectangular $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$

Polar $$\rho = \frac{ep}{1 + e\cos\theta}$$

Parabola $e = 1,$

$y^2 = 4ax \qquad x^2 = 4ay$

Distance, vertex to focus, a

Distance, focus to directrix, $p = 2a$

Latus rectum, length, $2p = 4a$

Central conics $e \neq 1$

Ellipse $e < 1,\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$ Standard Form

a = semimajor diameter

b = semiminor diameter

Foci $(\pm ae, 0)$

Hyperbola $e > 1,\ \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \pm 1$

a = semitransverse diameter

b = semiconjugate diameter

Foci $(\pm ae, 0)$

CHAPTER

VI

TRANSFORMATION OF AXES

VI.1 TRANSLATION OF AXES

We recall that in the study of the circle, the equation of the circle with the center at the origin had a simpler form than one with the center at (a,b). If, in working with a circle, we were able to change to a set of co-ordinate axes with the origin at the center of the curve, we might find a decided advantage from the algebraic point of view.

For example, consider the circle

(1) $$x^2 + y^2 - 4x - 6y - 12 = 0.$$

or

(2) $$(x - 2)^2 + (y - 3)^2 = 25.$$

Now imagine a new pair of axes, parallel to the x- and y-axes,

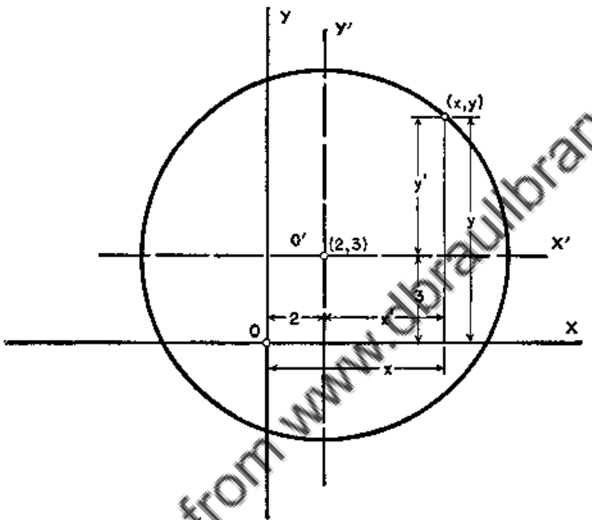

FIG. 57.—CHANGE OF AXES.

but passing through the point (2,3). As Fig. 57 shows,

(3) $$x' = x - 2,$$
$$y' = y - 3,$$

or

(4) $$x = x' + 2,$$
$$y = y' + 3.$$

When we substitute in Equation (2), we get

(5) $$x'^2 + y'^2 = 25.$$

This transformation to a new set of axes parallel to the original axes, but through a new origin, has for its purpose, as in this example, the reduction of an equation to a simpler form, which will make the analysis of the locus easier.

In general, a translation of axes to a new origin (x_0,y_0) is effected (see Fig. 58) by the substitution

$$(6)\qquad \begin{aligned} x &= x' + x_0,\\ y &= y' + y_0. \end{aligned}$$

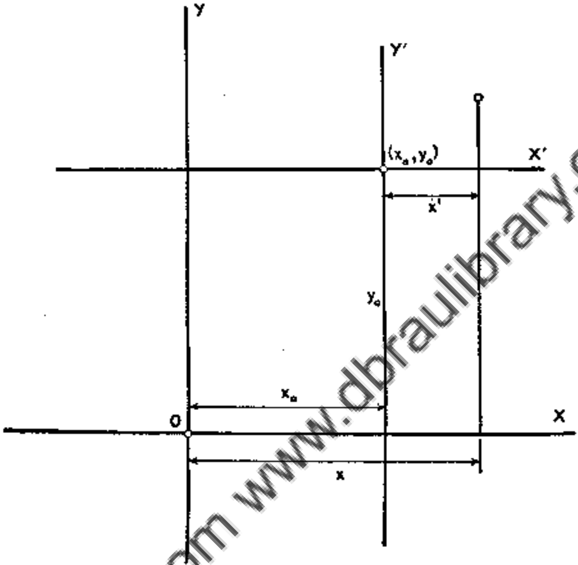

FIG. 58.—CHANGE OF AXES.

EXERCISES

1. Translate the axes to the new origin (4,2), and find the new co-ordinates of the points (1,2), (0,−3), (−4,5), (−1,8), (x,y).
2. When referred to axes through the point (−1,3), the new co-ordinates of four points are (3,4), (6,−1), (0,5), (0,0). What were their original co-ordinates?
3. By translating to the point (−1,0) as a new origin, find the new equations of the straight lines
   (a) $3x + 4y - 5 = 0$, (b) $y = 10x + 6$,
   (c) $ax + by + c = 0$.
4. Perform the translation to the indicated point as new origin, simplify the equation and identify the curve:
   (a) $(x-1)^2 + y^2 = 4$, (1,0).
   (b) $x^2 + y^2 + 4x - 2y - 4 = 0$, (−2,1).
   (c) $(y-1)^3 = 4(x-4)$, (4,1).
   (d) $x^2 + 6x - 8y + 33 = 0$, (−3,3).
   (e) $\dfrac{(x+3)^2}{25} + \dfrac{(y-2)^2}{16} = 1$, (−3,2).
   (f) $3x^2 + y^2 + 12x + 6y - 3 = 0$, (−2,−3).
   (g) $\dfrac{(x+1)^2}{4} - \dfrac{(y+3)^2}{32} = 1$, (−1,−3).

5. Determine the translation of axes that will reduce the following equations either to the form $y^2 = 4ax$ or to $x^2 = 4ay$ (try completing squares):
   (a) $y^2 - 6x + 4y + 10 = 0.$ (c) $x^2 - 6x + 10y - 41 = 0.$
   (b) $y^2 + 4x - 3y - 6 = 0.$ (d) $ax^2 + 2gx + 2fy + c = 0.$
6. Determine the translation that will reduce the following equations to the symmetric form (without first degree terms):
   (a) $4x^2 + 9y^2 + 8x - 32 = 0.$ (c) $3x^2 - y^2 - 6x - 2y - 10 = 0.$
   (b) $4x^2 + y^2 - 6y + 5 = 0.$ (d) $7x^2 - 9y^2 + 28x + 91 = 0.$
   (e) $ax^2 + by^2 + 2gx + 2fy + c = 0.$

In the following exercises, find the equation of the conic, sketch and determine the usual important points and lines. (Foci, directrices, etc.)

7. Vertex (4,1), focus (5,1), $e = 1$.
8. $e = 3/5$, center $(2,-1)$, principal diameter parallel to x-axis and of length 10.
9. Principal diameter = 6, foci (6,0), $(-2,0)$.
10. Vertex (4,1), focus (3,1), eccentricity 1.
11. $e = 3/4$, center (0,5), one directrix $3y - 31 = 0$.
12. Focus $(-3,5)$, directrix $y = 1$, $e = 1$.
13. $e = \sqrt{3}$, principal axis $x + 2 = 0$, center $(-2,1)$, principal diameter 2.
14. Vertex (h,k), $e = 1$, axis vertical, length of *latus rectum* $4a$, curve extends downward.
15. Parabola, vertex (h,k), axis horizontal, curve opens to left, *latus rectum* $= 4a$.
16. $e<1$, center (h,k), semidiameters a,b, principal axis vertical.
17. As in 16, but with principal axis horizontal.
18. (a) $e>1$, center (h,k), semimajor diameter $= a$, principal axis vertical.
    (b) As in (a) but with principal axis horizontal.

VI.2 FINDING THE CENTER OF A CONIC

The center of a conic is a center of symmetry. That is to say, any straight line through the center meets the conic in two points that are equally distant from the center. (See R,S in Fig. 59.)

The most general equation of the second degree in x and y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{1}$$

and we have seen that certain forms of this equation have conics for their loci.

If the curve is symmetrical with respect to the origin, i.e., if the center of the curve lies at the origin, changing the signs of both x and y will not alter the equation. That is, the equation

$$ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0 \tag{2}$$

must be identical with Equation (1). But this can only happen if the coefficients of the first-degree terms are zero.

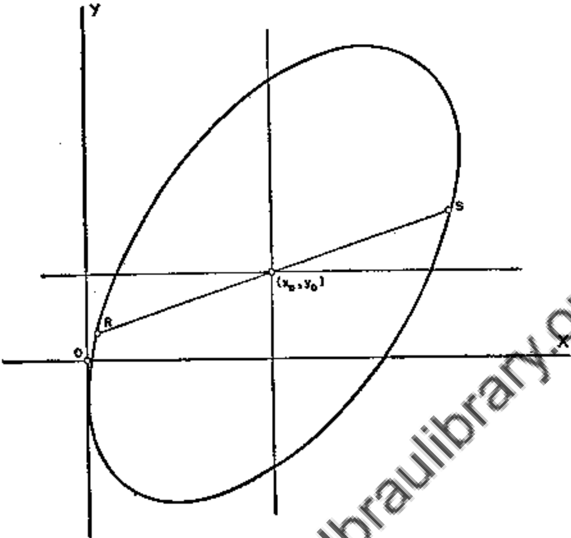

FIG. 59.

Let us suppose that the center of Curve (1) is the point (x_0,y_0). Transforming to this point as a new origin,

(3) $$x = x' + x_0,$$
$$y = y' + y_0.$$

Substituting in Equation (1), we have

(4) $$a(x' + x_0)^2 + 2h(x' + x_0)(y' + y_0) + b(y' + y_0)^2 + 2g(x' + x_0) + 2f(y' + y_0) + c = 0.$$

When we expand and collect terms we obtain

(5) $$ax'^2 + 2hx'y' + by'^2 + 2x'(ax_0 + hy_0 + g) + 2y'(hx_0 + by_0 + f) + x_0 (ax_0 + hy_0 + g) + y_0(hx_0 + by_0 + f) + (gx_0 + fy_0 + c) = 0.$$

Now if the new origin is the center of the curve, the first degree terms in x' and y' are missing. That is to say,

(6) $$ax_0 + hy_0 + g = 0,$$
$$hx_0 + by_0 + f = 0.$$

If we assume that

$$ab - h^2 \neq 0,$$

these two equations may be solved for the values of x_0,y_0. Then if these values are substituted in Equation (5), we have

(7) $$ax'^2 + 2hx'y' + by'^2 + C = 0, \quad \text{where}$$

$$C = \frac{abc - af^2 + 2fgh - ch^2 - bg^2}{ab - h^2}.$$

Hence a necessary condition that (2) may represent a central conic, that is, an ellipse or hyperbola, is $ab - h^2 \neq 0$. We shall see later that under certain conditions (2) may be transformed into one of the standard forms discussed in Chapter V.

EXAMPLE. Find the center of the conic

$$3x^2 - 8xy + y^2 + 2x + 6y + 13 = 0,$$

and find the equation referred to this point as origin.

Comparing with Equation (1) we see that

$$a = 3,\ h = -4,\ b = 1,\ g = 1,\ f = 3,\ c = 13.$$

Then the two equations determining the center are

$$\begin{aligned} 3x_0 - 4y_0 + 1 &= 0, \\ -4x_0 + y_0 + 3 &= 0. \end{aligned}$$

From these, on solving, we find $x_0 = 1$, $y_0 = 1$.
Then

$$C = gx_0 + fy_0 + c = 1 \cdot 1 + 3 \cdot 1 + 13 = 17.$$

The new equation, therefore, is

$$3x'^2 - 8x'y' + y'^2 + 17 = 0.$$

VI.3 PARABOLA

If, in trying to find the center of the conic as in §VI.2, it should happen that $ab - h^2 = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = 0$, we should not have any finite center for the conic. But the conic without a center is the parabola. Hence a necessary condition that the equation

(1) $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a parabola is that the second-degree terms,

(2) $$ax^2 + 2hxy + by^2,$$

form a perfect square.

EXAMPLE. Find the center of $x^2 - 4xy + 4y^2 - 8x - 4y + 8 = 0$.

The equations for the determination of the center of the conic are

$$x - 2y - 4 = 0,$$
$$-2x + 4y - 2 = 0.$$

Their graphs are parallel lines, hence there is no center.

The second-degree terms form a perfect square, and it may be shown that the curve is a parabola.

VI.4 DEGENERATE CONICS

It might happen that $c = 0$ in (7) of VI.3, that is,

$$abc + 2fgh - af^2 - ch^2 - bg^2 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

(1) Then $$ax'^2 + 2hx'y' + by'^2 = 0.$$

But the left member of this equation can be broken up into two linear factors, as follows:

$$ax'^2 + 2hx'y' + by'^2 = \frac{1}{a}[a^2x'^2 + 2ahx'y' + h^2y'^2 - (h^2 - ab)y'^2]$$
$$= \frac{1}{a}[ax' + [h - \sqrt{h^2 - ab}]y']$$
$$[ax' + (h + \sqrt{h^2 - ab})]y'] = 0.$$

The original equation represents a locus consisting of the two straight lines

$$a(x - x_0) + (h - \sqrt{h^2 - ab})(y - y_0) = 0,$$
$$a(x - x_0) + (h + \sqrt{h^2 - ab})(y - y_0) = 0,$$

and the locus is said to be a degenerate conic. Both of these lines pass through the center (x_0,y_0). If $h^2 - ab < 0$, so that the lines are imaginary, the point (x_0,y_0) is the only point with real co-ordinates lying on either line. Some people call such a conic a **point conic.** We can then say:

The condition that the locus of

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

consists of two straight lines, or is a degenerate conic, is that

the determinant, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$

EXAMPLE. Discuss the equation $2x^2 + 2xy - 2y^2 - 10x + 10 = 0.$

To find the center, we solve

$$\begin{aligned} 2x_0 + y_0 - 5 &= 0, \\ x_0 - 2y_0 &= 0, \end{aligned}$$

and find (2,1).

The reduced equation is

$$2x'^2 + 2x'y' - 2y'^2 = 0, \text{ since } \begin{vmatrix} 2 & 1 & -5 \\ 1 & -2 & 0 \\ -5 & 0 & 10 \end{vmatrix} = 0.$$

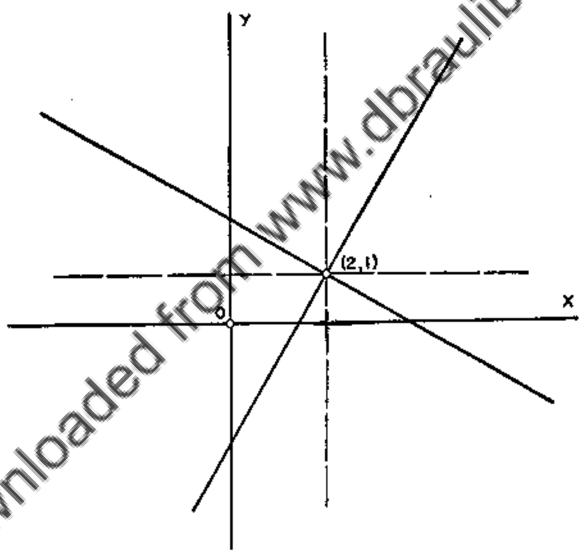

FIG. 60.

The locus (see Fig. 60) consists of the two straight lines

$$\begin{aligned} 2x' + (1 - \sqrt{5})y' &= 0, \\ 2x' + (1 + \sqrt{5})y' &= 0, \end{aligned}$$

or, in the original co-ordinates,

$$\begin{aligned} 2x + (1 - \sqrt{5})y - 5 + \sqrt{5} &= 0, \\ 2x + (1 + \sqrt{5})y - 5 - \sqrt{5} &= 0. \end{aligned}$$

EXERCISES

1. Show the steps in deriving Equation (5) of VI.2.
2. Show all the steps in deriving (7) and (8) of VI.2.

In Exercises 3–8, find the center of the conic, test for degeneracy, and write the equation after translating to the center as origin.

3. $4x^2 + 10xy + 4y^2 + 15 = 0$.
4. $4x^2 + 10xy + 4y^2 = 0$.
5. $x^2 - 4xy + 5y^2 - 8x + 6y + 43 = 0$.
6. $x^2 - 4xy + 5y^2 - 8x + 6y + 41 = 0$.
7. $4x^2 - 7xy + 3y^2 - 2x + 4y - 9 = 0$.
8. $2x^2 - 5xy - 3y^2 - x + 3y = 0$.
9. Show that the following conics are parabolas, and that (b) and (c) are degenerate (consist of two parallel straight lines). Find the equations of these straight lines.
   (a) $x^2 + 2xy + y^2 + 15x - 13y + 60 = 0$.
   (b) $x^2 + 2xy + y^2 + x + y - 2 = 0$.
   (c) $x^2 + 4xy + 4y^2 - 6x - 12y + 9 = 0$.

In Exercises 10–16, find the center, transform the origin to the center, and identify the curve.

10. $x^2 + y^2 - 10x + 4y + 22 = 0$.
11. $x^2 + 2y^2 - 4x + 2y - 12 = 0$.
12. $x^2 + y^2 - 10x + 4y + 29 = 0$.
13. $3x^2 + 4y^2 - 18x - 8y + 19 = 0$.
14. $9y^2 - 2x^2 + 12x - 36 = 0$.
15. $x^2 + y^2 - 10x + 4y + 39 = 0$.
16. $x^2 - y^2 - 2x - 5 = 0$.

VI.5 ASYMPTOTES

Let us consider the problem of finding points on the curve whose equation is

(1) $$3x^2 - 8xy - 3y^2 = 4.$$

Perhaps as easy a method as any is to choose an arbitrary line through the origin,

(2) $$y = mx,$$

substitute in Equation (1), and solve for x, then for y.

(3) $$(3 - 8m - 3m^2)x^2 = 4.$$

(4) $$x = \frac{\pm 2}{\sqrt{3 - 8m - 3m^2}} = \frac{\pm 2}{\sqrt{(3 + m)(1 - 3m)}}.$$

By inserting values for m, the values for x and y are easily computed. Some general considerations may shorten the labor.

For values of m between -3 and $+1/3$, both factors under the radical are positive, and hence we find real points on the curve. For values of m outside of this range, one factor will be positive, the other negative, and therefore the points will be impossible to plot, since they have imaginary co-ordinates.

We note that as m comes close to 1/3, or to -3, the values of x are very large. The corresponding lines for $m = 1/3$ and -3,

$$(5) \qquad \begin{aligned} 3x + y &= 0, \\ x - 3y &= 0, \end{aligned}$$

are called **asymptotes** of the curve (Fig. 61).

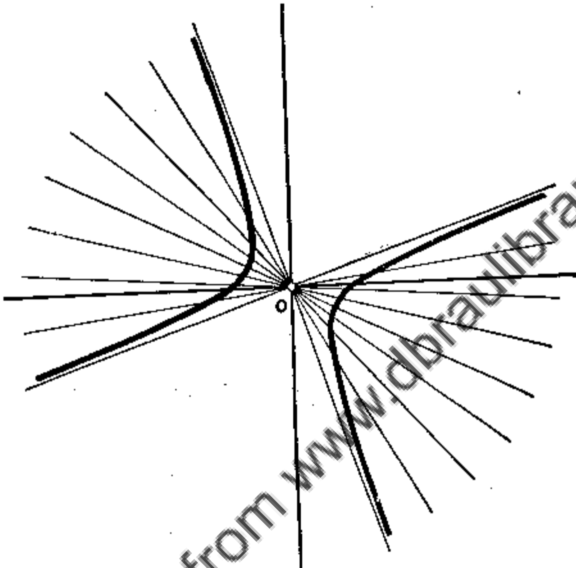

FIG. 61.—ASYMPTOTES OF A HYPERBOLA.

m	x
-2.75	± 1.32
-2.50	± 1.03
-2.25	$\pm .83$
-2.00	$\pm .76$
-1.75	$\pm .72$
-1.50	$\pm .70$
-1.25	$\pm .69$
-1.00	$\pm .71$
$-.75$	$\pm .74$
$-.50$	$\pm .80$
$-.25$	$\pm .91$
-0.00	± 1.16
$.25$	± 1.43
$.30$	± 3.5

The asymptotes of a hyperbola are of particular interest, since they serve as guide lines in sketching the curve.

It is interesting to see that if we multiply the left members of Equations (5) together, we have the left member of Equation (1).

It can be shown, in general, that if the center of the conic lies at the origin, one may find the asymptotes by factoring the second-degree terms and setting each factor equal to zero. The reasoning follows very closely the reasoning in the foregoing illustrative example. (See Exercise 1.)

Thus, to find the asymptotes of the curve

$$(6) \qquad ax^2 + 2hxy + by^2 + c = 0,$$

we set

$$(7) \qquad ax^2 + 2hxy + by^2 = 0,$$

factor, and set each factor equal to zero.

Since an ellipse is a finite closed curve, there will be no real asymptotes. Hence for an ellipse

$$(8) \qquad h^2 - ab < 0.$$

If the curve is a hyperbola, the asymptotes are real, and, for that case,

$$(9) \qquad h^2 - ab > 0.$$

As we have already seen, that for a parabola is

$$(10) \qquad h^2 - ab = 0.$$

Thus we have a very convenient means of determining the type of curve without the labor of reducing the equation to standard form. If the locus of the general equation is not a degenerate conic and if

(11) $h^2 - ab > 0$, the curve is a hyperbola.
$h^2 - ab = 0$, the curve is a parabola.
$h^2 - ab < 0$, the curve is an ellipse.

EXAMPLE 1. Sketch the curve

$$9x^2 - 4y^2 = 36.$$

The curve is a hyperbola (Fig. 62), since

$$h^2 - ab = 0 - 9(-4) = 36 > 0.$$

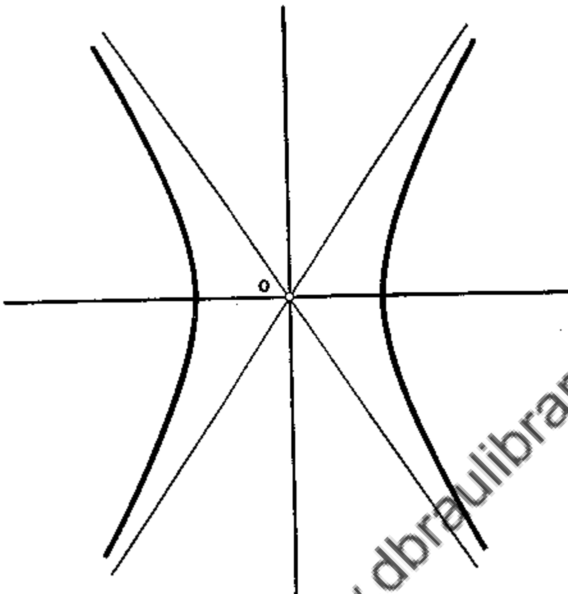

FIG. 62.—EXAMPLE 1.

Its asymptotes are the lines $3x + 2y = 0$, and $3x - 2y = 0$.

Its vertices are the points (2,0), (−2,0).

EXAMPLE 2. Sketch the curve $xy = 1$. The curve is symmetrical with respect to the origin and the line $y = x$. Its asymptotes are the lines $x = 0$, $y = 0$. The points (1,1), (−1,−1), are points on the curve. (Fig. 63.)

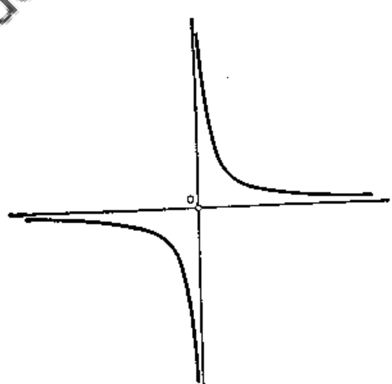

FIG. 63.—EXAMPLE 2.

EXERCISES

1. Given the conic $ax^2 + 2hxy + by^2 + c = 0$.
   (a) Find the intersections with the line $y = mx$.
   (b) Show that x and y are no longer finite when
   $$m = \frac{-h \pm \sqrt{h^2 - ab}}{b}.$$
   (c) Show that the lines corresponding to these values of m are
   $$(-h + \sqrt{h^2 - ab})x - by = 0,$$
   $$(-h - \sqrt{h^2 - ab})x - by = 0.$$
   (d) Show that the product of the left members of these equations leads to
   $$b(ax^2 + 2hxy + by^2) = 0,$$
   and hence that to find the asymptotes we equate the second degree terms to zero.

In Exercises 2–7, identify the given curve, find the equations of the asymptotes, and sketch them if they are real.

2. $9x^2 - y^2 - 36 = 0$. **Ans.** $y = 3x$, $y = -3x$.
3. $\frac{x^2}{4} - y^2 = 1$. **Ans.** $y = \frac{x}{2}$, $y = -\frac{x}{2}$.
4. $x^2 + 4y^2 - 16 = 0$.
5. $b^2x^2 - a^2y^2 - a^2b^2 = 0$.
6. $5x^2 - 6xy + y^2 - 9 = 0$. **Ans.** $x - y = 0$, $5x - y = 0$.
7. $3x^2 - 4xy + y^2 + 2 = 0$.
8. Prove that the asymptotes of the equilateral hyperbola, $x^2 - y^2 = a^2$ are perpendicular.

In the following exercises, find the center of the curve, translate to the center as a new origin, identify the curve, find the equations of the asymptotes, and translate back to the original co-ordinate system.

9. $xy + y^2 + 2x - 6y = 0$.
10. $x^2 - 2y^2 - 4x - 2y + 1 = 0$. **Ans.** Center, $\left(2, -\frac{1}{2}\right)$, hyperbola,
$$x - y\sqrt{2} - \left(2 + \frac{1}{2}\sqrt{2}\right) = 0,$$
$$x + y\sqrt{2} - \left(2 - \frac{1}{2}\sqrt{2}\right) = 0.$$
11. $x^2 + 2xy + 2y^2 - 8y + 1 = 0$.

VI.6 ROTATION OF AXES

If there is no xy-term in the equation of a conic, the methods previously discussed enable us to make a fairly complete analysis of the curve. Any equation that contains an xy-term can be reduced to the other form by a rotation of axes.

Suppose a new set of axes with the same origin is found by rotating the old axes through an angle θ (Fig. 64). Then the

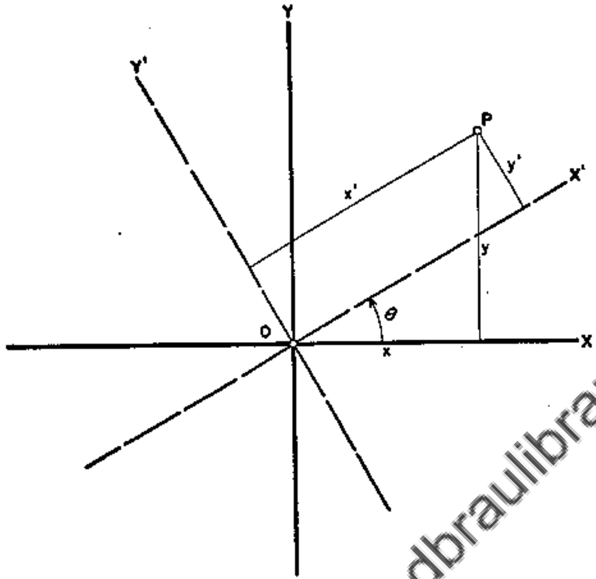

FIG. 64.—ROTATION OF AXES.

equation of the new y-axis is

(1) $$x \cos \theta + y \sin \theta = 0,$$

and the equation of the new x-axis is

(2) $$-x \sin \theta + y \cos \theta = 0.$$

The new x is the distance of the point (x,y) from the line (1), and the new y is the distance of (x,y) from the line (2). Therefore, by applying the properties of the normal form of the straight line, we get the equations of the transformation,

(3) $$\begin{aligned} x' &= x \cos \theta + y \sin \theta, \\ y' &= -x \sin \theta + y \cos \theta, \end{aligned}$$

or, upon solving for x and y,

(4) $$\begin{aligned} x &= x' \cos \theta - y' \sin \theta, \\ y &= x' \sin \theta + y' \cos \theta. \end{aligned}$$

Now if we apply this transformation to the equation

(5) $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the new xy-term has the coefficient

(6) $$-a \cdot 2\sin \theta \cos \theta + 2h(\cos^2 \theta - \sin^2 \theta) + b \cdot 2\sin \theta \cos \theta.$$

If we set this equal to zero and apply the formulas of trigonometry, we find

$$\tan 2\theta = \frac{2h}{a - b}. \tag{7}$$

From this, it is easy to determine $\cos 2\theta$, and, by the use of the formulas

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}}, \qquad \cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}}, \tag{8}$$

the equations of the transformation can be found.

EXAMPLE. By a rotation of axes, eliminate the xy-term from the equation $3x^2 + 24xy - 4y^2 - 156 = 0$.

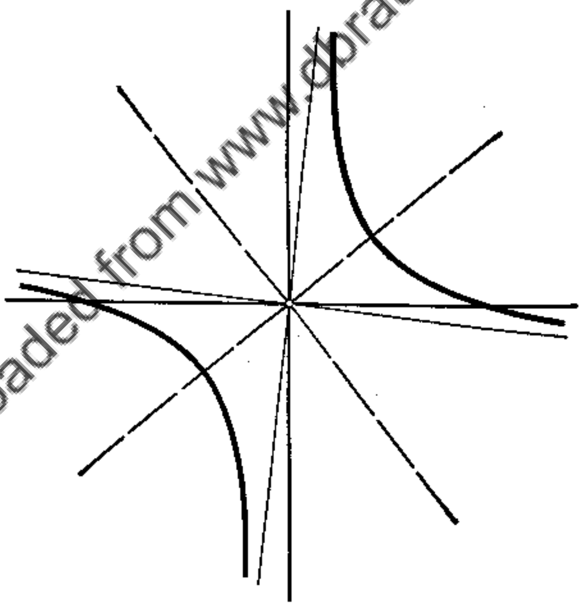

FIG. 65.

$$\tan 2\theta = \frac{24}{7}, \cos 2\theta = \frac{7}{25},$$

$$\sin \theta = \frac{3}{5}, \cos \theta = \frac{4}{5}.$$

Hence $$x = \frac{4}{5}x' - \frac{3}{5}y', \qquad y = \frac{3}{5}x' + \frac{4}{5}y'.$$

Substituting in the equation, we have

$$\frac{3}{25}(16x'^2 - 24x'y' + 9y'^2) + \frac{24}{25}(12x'^2 + 7x'y' - 12y'^2)$$
$$-\frac{4}{25}(9x'^2 + 24x'y' + 16y'^2) - 156 = 0.$$

When we collect terms, this becomes

$$12x'^2 - 13y'^2 - 156 = 0.$$

If the conic is a central conic, we usually translate the axes to the center as a new origin before rotating the axes. If the curve is a parabola, we proceed to rotate the axes at once.

Since rotation of axes does not change the form of a curve, we are now in position to verify our assumption that the general equation of the second degree in x and y is either a conic or a degenerate conic. It is always possible to remove the xy-term and the resulting equation may then be reduced to one of the standard forms of a conic or to a degenerate form.

EXERCISES

1. What property of the normal form of the straight line is used in deriving the equations for rotation of axes?
2. Derive Equations (4) from Equations (3).
3. Show the derivation of cos 2θ from tan 2θ.

Eliminate the xy-term, identify the conic, and sketch on the new set of axes:

4. $13x^2 + 12xy + 8y^2 - 68 = 0$.
5. $2x^2 + 5xy + 2y^2 - 18 = 0$.
6. Eliminate the xy-term, and show that $5x^2 + 3xy + y^2 = 0$ is a degenerate ellipse.

Reduce to standard form and identify the conic:

7. $3x^2 - 5xy + 3y^2 + x + 11y - 1 = 0$.
8. $2x^2 - 4xy + 5y^2 + 8x + 6y + 41 = 0$.
9. $3x^2 + 8xy - 3y^2 + 10x + 10 = 0$.
10. Reduce to standard form, draw the sets of axes, and, on the last, sketch the conic
$$2x^2 - 4xy - y^2 - 6y + 15 = 0.$$

Reduce to standard form:

11. $4x^2 - 12xy + 9y^2 + 3x + 2y - 13 = 0$.
12. $x^2 + 4y^2 + 4xy - 6x + 8y = 0$.
13. $x^2 - 2xy + y^2 - 4x - 4y + 8 = 0$.
14. Reduce to standard form and sketch the curve
$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0.$$

REVIEW

1. On the same set of co-ordinate axes, construct the three lines
   (a) with slope $= -\frac{1}{2}$, passing through $(1,4)$,
   (b) with x-intercept $= 3/2$, y-intercept $= -3$,
   (c) with $p = 3\sqrt{5}$, $\omega = \text{arc tan }(-2)$.
2. Find the equations of the lines of Exercise 1, and the co-ordinates of the vertices of the triangle formed by them.
3. Show that for the triangle of Exercises 1 and 2, the equation of the escribed circle, tangent to (b) and to the extensions of (a) and (c), is $(x - 12)^2 + (y - 6)^2 = 45$. Draw the circle.
4. Without first drawing, determine the relative positions of the circles $x^2 + y^2 = 16$, and $x^2 + y^2 - 2x - 16y + 61 = 0$.
5. Draw the following circles and show that they have one point in common:

$$x^2 + y^2 - 10x - 14y + 54 = 0,$$
$$x^2 + y^2 - 4x - 17y + 45 = 0,$$
$$x^2 + y^2 - 9y + 9 = 0.$$

6. A point moves so that its distance from the point $(-2,3)$ is always 1 less than its distance from the line $x - 4 = 0$. Obtain the equation of the locus, identify, and sketch.
7. Show that the equation of the locus of points whose distance from the point $(3,0)$ is twice its distance from the y-axis is $3x^2 - y^2 + 6x - 9 = 0$. Identify and sketch, showing the important points and lines.
8. Find the center of the conic $18x^2 + 24xy + 25y^2 + 36x - 10y + 1 = 0$. Transform to the center as a new origin, eliminate the xy-term, and reduce to standard form.

SUMMARY OF CHAPTER VI

Translation of axes

$$\begin{cases} x' = x - x_0 \\ y' = y - y_0 \end{cases} \qquad \begin{cases} x = x' + x_0 \\ y = y' + y_0 \end{cases}$$

Center of conic $\quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$\begin{aligned} ax_0 + hy_0 + g &= 0 \\ hx_0 + by_0 + f &= 0 \end{aligned}$$

where $ah - h^2 \neq 0$

Degenerate conic

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Asymptotes of hyperbola $\quad ax^2 + 2hxy + by^2 + c = 0$

$$ax^2 + 2hxy + by^2 = 0$$

Rotation of axes

$$\begin{cases} x' = x \cos\theta + y \sin\theta \\ y' = -x \sin\theta + y \cos\theta \end{cases} \qquad \begin{cases} x = x' \cos\theta - y' \sin\theta \\ y = x' \sin\theta + y' \cos\theta \end{cases}$$

To eliminate xy-term from general conic

$$\tan 2\theta = \frac{2h}{a - b}, \quad \sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}}, \quad \cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}}$$

CHAPTER

VII

TANGENTS TO CONICS

VII.1 INTRODUCTION—TANGENT WITH GIVEN SLOPE

A straight line may cut a conic in two distinct points. The coordinates of the points of intersection are the simultaneous solutions of the equations of the line and the conic.

A straight line is said to be tangent to a conic if the line's two points of intersection with the conic coincide. The condition for this is, as we have seen in the case of the circle, that when one variable is eliminated between the equations of line and curve, the discriminant of the resulting quadratic equation vanishes.

The determination of tangents to a curve from an outside point, or of tangents having a given slope, depends upon the application of this condition.

EXAMPLE 1. Determine the tangents to the curve $x^2 + 2y^2 - 4x = 0$ that pass through the point $(6, \sqrt{2})$.

The line with slope m, passing through $(6,\sqrt{2})$, is

$$y = mx - 6m + \sqrt{2}.$$

Substituting in the equation of the conic and collecting terms, we have

$$(1 + 2m^2)x^2 - (24m^2 - 4m\sqrt{2} + 4)x + (72m^2 - 24m\sqrt{2} + 4) = 0.$$

Finding the discriminant, $b^2 - 4ac$, and removing a common numerical factor, we have the equation

$$-3m^2 + 2m\sqrt{2} = 0.$$

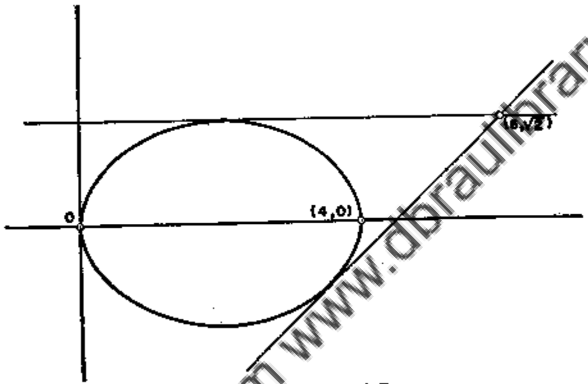

FIG. 66.—EXAMPLE 1.

Whence the necessary slopes are 0 and $\frac{2}{3}\sqrt{2}$. The desired tangents (Fig. 66) are

$$y = \sqrt{2}, \text{ and } 3y - 3\sqrt{2} = 2x\sqrt{2} - 12\sqrt{2}.$$

EXAMPLE 2. Determine the tangent to the curve $x^2 - 4y - 4 = 0$ that has the slope 2.

The equation of any straight line having the slope 2 is

$$y = 2x + b.$$

Substituting this value for y in the equation of the conic we have

$$x^2 - 8x - 4b - 4 = 0.$$

This equation will have equal roots provided the discriminant vanishes, i.e., if

$$16 + 4b + 4 = 0,$$
$$b = -5.$$

Hence the equation of the desired tangent (Fig. 67) is

$$y = 2x - 5.$$

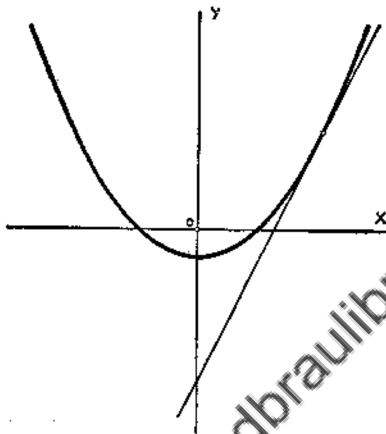

FIG. 67.—EXAMPLE 2.

EXERCISES

Determine the equations of the tangents to each of the following through the indicated points.

1. $x^2 + y^2 = 25$, $(7,1)$.
2. $9x^2 + 16y^2 = 144$, $\left(1, \dfrac{21}{5}\right)$.
3. $y^2 = 8x - 7$, $(-1,-1)$.
4. $5x^2 - 2y^2 = 18$, $(1,-4)$.
5. $y^2 - 3x - 6y + 21 = 0$, $(1,3)$.

Determine tangents to each of the following with the indicated slopes.

6. $x^2 + y^2 = 16$, slope $= -\dfrac{4}{3}$.
7. $x^2 + y^2 - 8x + 4y = 0$, slope $= 2$.
8. $y^2 + 24x = 0$, slope $= -3$.
9. $9x^2 + 16y^2 = 144$, slope $= -\dfrac{1}{4}$.
10. $36x^2 - 25y^2 = 900$, slope $= 2$.
11. Show that the equation of the tangent to the parabola $y^2 = 4px$ with slope m is $y = mx + \dfrac{p}{m}$.
12. Show that the equations of the tangents to the circle $x^2 + y^2 = r^2$ with slope m are $y = mx \pm r\sqrt{m^2 + 1}$.
13. Show that the equations of the tangents to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with slope m are $y = mx \pm \sqrt{a^2m^2 + b^2}$.
14. Show that the equations of the tangents to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with slope m are $y = mx \pm \sqrt{a^2m^2 - b^2}$.

15. Determine the equations of the tangents to the circle $x^2 + y^2 = 100$ that are parallel to the line $4x - 3y = 20$.

16. Find the equation of the tangent to the parabola $x^2 = 12y$ that is perpendicular to the line $x - 3y + 2 = 0$.

17. Prove that if a real tangent can be drawn to the parabola
$$y = ax^2 + bx + c \ (a>0)$$
through the point (h,k), then
$$ah^2 + bh + c > k.$$

18. Show that the line
$$x \cos \omega + y \sin \omega - p = 0$$
is tangent to the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
if
$$p^2 = a^2 \cos^2 \omega + b^2 \sin^2 \omega.$$

VII.2 TANGENT TO A CONIC AT A POINT

Consider the general conic

(1) $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If the point (x_1,y_1) lies on this curve,

(2) $$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0.$$

Any straight line through this point generally meets the curve

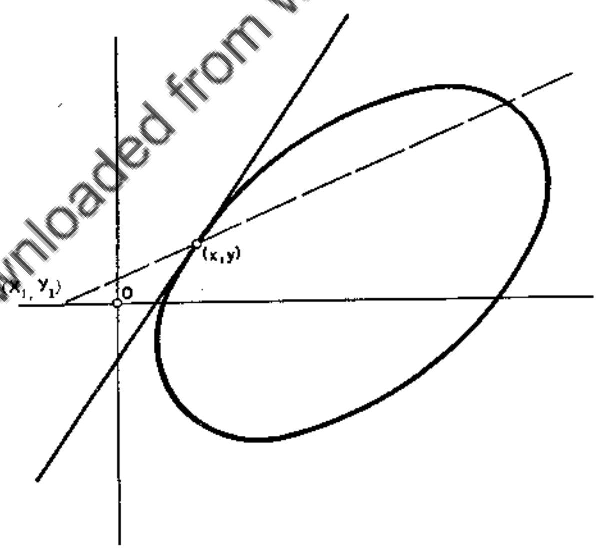

FIG. 68.—TANGENT TO A CONIC.

in two points, the point (x_1,y_1) and another point (Fig. 68).

To find the second point, we take the equation of the straight line,

$$(3) \qquad y - y_1 = m(x - x_1),$$

simultaneously with the equation of the conic, (1), and solve.

Subtracting Equation (2) from Equation (1), we have

$$(4) \quad a(x^2 - x_1^2) + 2h(xy - x_1y_1) + b(y^2 - y_1^2) + 2g(x - x_1) + 2f(y - y_1) = 0.$$

If we eliminate y from Equations (3) and (4), the resulting form can be factored into

$$(5) \quad (x - x_1)(ax + 2hmx + bm^2x + ax_1 + 2hy_1 + 2g + 2fm - bm^2x_1 + 2by_1m) = 0.$$

If the line (3) is tangent, the second factor must vanish when x_1 is substituted in place of x, since the tangent meets the curve in two coincident points. Hence,

$$(6) \qquad 2ax_1 + 2hy_1 + 2g + m(2hx_1 + 2by_1 + 2f) = 0.$$

Since m must satisfy this equation if the line (3) is to be a tangent, the slope of the tangent must be

$$(7) \qquad m = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}.$$

The equation of the tangent is then,

$$(8) \quad (ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0.$$

If we expand and rearrange terms, we get the equation

$$(9) \qquad axx_1 + hxy_1 + hx_1y + byy_1 + gx + gx_1 + fy + fy_1 - (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1) = 0.$$

From Equation (2), it is readily seen that the second line in Equation (9) is equal to c, and hence that the tangent at the point (x_1,y_1) on the conic is,

$$(10) \quad axx_1 + hxy_1 + hx_1y + byy_1 + gx + gx_1 + fy + fy_1 + c = 0.$$

This equation is sometimes written in the form

$$(11) \quad (ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + (gx_1 + fy_1 + c) = 0.$$

EXAMPLE. Find the equation of the tangent to the curve $x^2 - 4xy - 5y^2 + 6x + 6y - 9 = 0$, at the point (2,1).

$$a = 1,\ h = -2,\ b = -5,\ g = 3,\ f = 3,\ c = -9.$$

The desired tangent (Fig. 69) is

$$(2 - 2 + 3)\, x + (-4 - 5 + 3)\, y + (6 + 3 - 9) = 0,$$

or

$$x - 2y = 0.$$

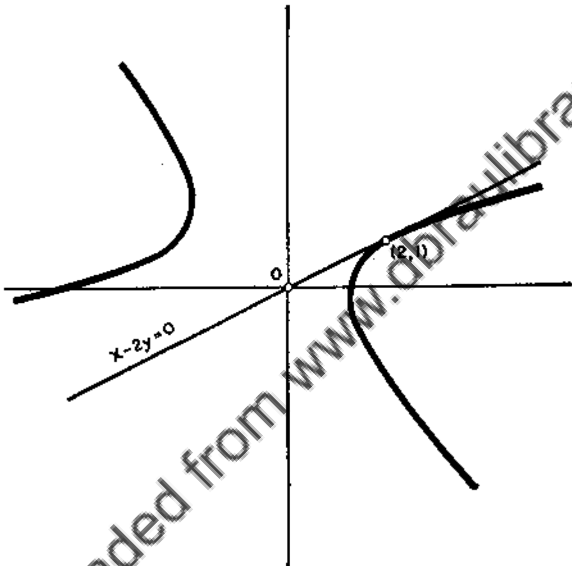

FIG. 69.

EXERCISES

1. (a) Show how Equation (5) is developed from (4).
   (b) Show how Equation (6) is found.
   (c) Express, in your own words, a method by which Equation (10) can be remembered easily when compared with Equation (1).
   (d) Compare Equation (11) with the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, and state a method of writing the equation of the tangent at a point of the curve.

The **normal** to a curve is the line perpendicular to the tangent at the point of contact. Hence its slope is minus the reciprocal of the tangent's slope. Find the equations of the tangent and normal to each of the following curves at the indicated points.

2. $x^2 + 2xy + y^2 - x + 2y - 12 = 0$, (1,2).
3. $x^2 + 6xy - 4y^2 + 8x + 15 = 0$, $(-3,0)$.

4. $9x^2 + 4y^2 + 12x + y - 65 = 0$, $(-1,4)$.
5. $y^2 - 4y + 2x - 1 = 0$, $(2,3)$.
6. $4x^2 - y^2 + 2y - x + 15 = 0$, $(0,5)$.
7. $x^2 + 6x + 10y - 10 = 0$, $(-6,1)$.
8. $9x^2 + 4y^2 - 18x + 16y - 11 = 0$, $(1,1)$.

Find the equations of the tangent and normal at the point (x_1,y_1) on the given curve in each of the following.

9. $x^2 + y^2 = r^2$.
10. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
11. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
12. $y^2 = 4ax$.
13. Given the parabola $y^2 = 4ax$. Prove,
    (a) the tangents at the ends of the *latus rectum* are perpendicular to each other.
    (b) the tangent at one end of the *latus rectum*, the perpendicular from the focus to this tangent, and the y-axis are concurrent.
    (c) the tangent at any point of the parabola, the perpendicular from the focus to this tangent, and the tangent at the vertex are concurrent.
14. Prove that the tangent at a point (x_1,y_1) on the parabola $y^2 = 4ax$ is the bisector of the angle between the focal radius to the point, and the line through the point parallel to the axis of the curve.

(It may be of some interest to note that the property of the parabola developed in Exercise 14 is utilized in the construction of the reflecting telescope, which reflects parallel light rays through a fixed point, the focus of the mirror. It is also used in reverse fashion in the construction of powerful searchlights.)

15. (a) Plot the ellipse $x^2/25 + y^2/16 = 1$, its foci F_1 and F_2, and the point $P(4,12/5)$.
    (b) Find the equations of the tangent and normal at P.
    (c) Find the equations of the focal radii, F_1P, F_2P.
    (d) Find the equation of the bisector of angle F_1PF_2.
    (e) Compare with the equations of tangent and normal.
16. Prove that the tangent and normal at any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$, bisect the angles formed by the focal radii to the point.

SUMMARY OF CHAPTER VII

Tangent to conic with slope m

Substitute $y = mx + k$. Set discriminant $= 0$. Solve for k

Tangent at point (x_1,y_1) on curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
$$ax_1x + hx_1y + hxy_1 + by_1y + g(x + x_1) + f(y + y_1) + c = 0$$

CHAPTER

VIII

CURVE TRACING IN POLAR CO-ORDINATES

VIII.1 CIRCLE AND STRAIGHT LINE

For the study of relationships that involve the trigonometric functions, we find that polar co-ordinates are often more convenient than the rectangular. Hence it is well for us to be able to graph equations in polar as well as in rectangular co-ordinates.

As we have seen in an earlier chapter (§IV.1), the equation

(1) $$\rho = a$$

represents a circle of radius a, with center at the pole. Also that the equation $\theta = \alpha$ represents a straight line through the pole.

Any other straight line has a polar equation (§III.2)

(2) $$\rho = p \sec(\theta - \alpha).$$

Any circle passing through the pole has an equation of form

(3) $$\rho = 2a \cos(\theta - \alpha).$$ (§IV.1.)

VIII.2 CONICS

We recall that the equation of a conic with focus at the pole and directrix perpendicular to the polar axis is

(1) $$\rho = \frac{ep}{1 \pm e \cos \theta},$$ (§V.3.)

the sign in the denominator being + or − according as the directrix is to the right or left of the pole.

EXERCISES

Identify and draw the following curves.

1. $\rho = 3.$
2. $\theta = \frac{\pi}{4}.$
3. $\theta = 0.$
4. $\theta = \frac{\pi}{2}.$
5. $\rho = 2 \sec\left(\theta - \frac{\pi}{6}\right).$
6. $\rho = 2 \sec \theta.$
7. $\rho = \csc \theta.$
8. $\rho = 4 \cos \left(\theta - \frac{\pi}{3}\right).$
9. $\rho = \sin \theta.$
10. $\rho = 8 \sin (\theta + \text{arc tan } 1).$
11. Write the polar equations of the conics with the following data:

(a) $e = 1, p = 3.$ (b) $e = \frac{3}{4}, p = 5.$ (c) $e = \frac{3}{2}, p = 4.$

Reduce the following equations to standard form, identify, and sketch.

12. $\rho = \frac{5}{2(1 + \cos \theta)}.$
13. $\rho = \frac{4}{1 - \cos \theta}.$
14. $\rho = \frac{12}{3 + 2 \cos \theta}.$
15. $\rho = \frac{20}{4 + 5 \cos \theta}.$

VIII.3 THE CISSOID

Frequently the polar equation offers a valuable suggestion for drawing a curve, as is the case with the cissoid (Fig. 70), which was first studied in the attempt to "double the cube." Its equation in rectangular form is

(1)
$$y^2 = \frac{x^3}{a - x}.$$

In polar co-ordinates the equation is obtained by substituting $x = \rho \cos \sigma$, $y = \rho \sin \sigma$ in (1). The resulting equation may be reduced to

(2)
$$\rho = a \sec \theta - a \cos \theta.$$

Then, in order to find points on the curve, we have merely to draw the straight line,

(3)
$$\rho = a \sec \theta,$$

and the circle,

(4)
$$\rho = a \cos \theta,$$

and subtract vectors. (Fig. 70: $OP = OL - OC$, $PL = OC$.) This can be done much more rapidly and accurately than by substituting values and computing the co-ordinates of the points.

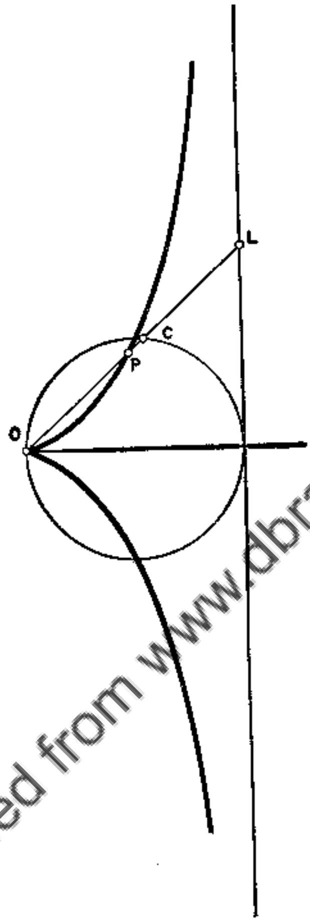

FIG. 70.—THE CISSOID.

VIII.4 THE LIMAÇON

Like the cissoid, the equation of the limaçon (Fig. 71), of which the cardioid is a special case, suggests a method of drawing.

(1) $$\rho = a \cos \theta + b.$$

We have merely to draw the circle,

(2) $$\rho = a \cos \theta,$$

and measure a distance b in the positive direction along the radius vector from the point on the circle. When $b = a$, the curve is the cardioid.

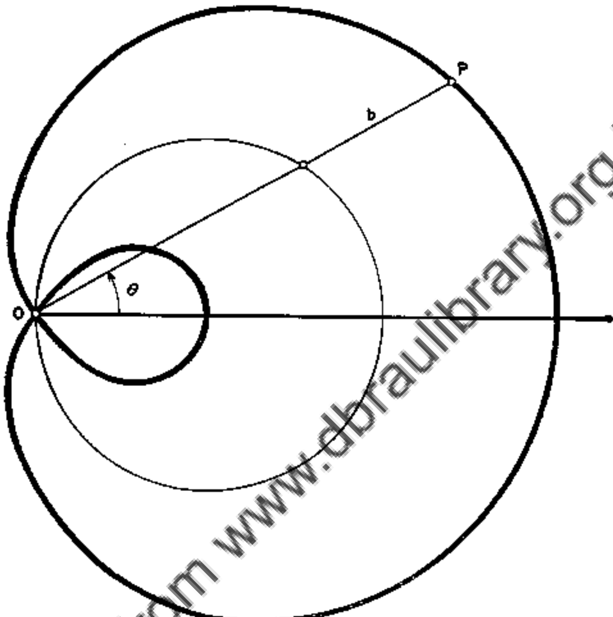

FIG. 71.—THE LIMAÇON.

EXERCISES

1. Name and sketch the curve

$$\rho = 2 \sec \theta - 2 \cos \theta.$$

2. Construct the cissoid

$$\rho = 4 \sec \theta - 4 \cos \theta.$$

3. (a) Sketch the curve

$$\rho = 2 \sec \theta + 2 \cos \theta.$$

(b) How does the drawing of 3(a) compare with that of Exercise 1?

Name and trace the following curves.

4. $\rho = 4 \cos \theta + 1$.
5. $\rho = 2 \cos \theta + 3$.
6. $\rho = a \cos \theta + a$.
7. $\rho = a \cos \theta + \frac{1}{4}a$.
8. $\rho = a \cos \theta + 2a$.
9. $\rho = 1 + \cos \theta$.
10. $\rho = 1 - \cos \theta$.
11. $\rho = 4 \cos \theta - 1$.
12. Sketch $\rho = a \sec \theta + b$. (Conchoid of Nicomedes.)

letting (a) $b = a$, (b) $b = \frac{a}{2}$, (c) $b = \frac{3a}{2}$.

13. Sketch $\rho = a \sec \theta - 2a \cos \theta$. (Strophoid.)

VIII.5 THE FOUR-LEAF ROSE

Of the curves representing functions of multiple angles two are of especial interest:

(1) $$\rho = a \cos 2\theta, \quad \rho = a \sin 2\theta.$$

As one can see from the properties of the functions, these two curves differ in only one respect. If the angle θ in the first is increased by $\pi/4$, the values of ρ are those of the second equation. Thus, having drawn one of these curves, we have merely to rotate the figure through an angle of 45° to get the other.

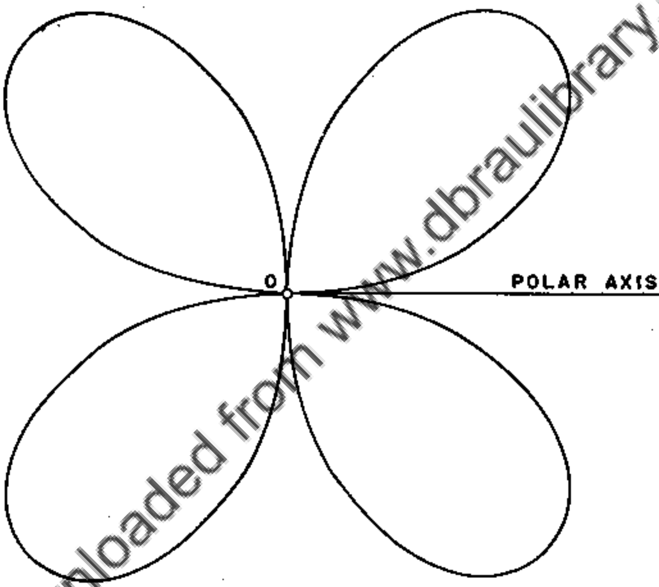

FIG. 72.—FOUR-LEAF ROSE.

Consider the curve (Fig. 72)

(2) $$\rho = a \sin 2\theta.$$

$$\rho \text{ is zero for } \theta = 0, \frac{\pi}{2}, \text{ and every multiple of } \frac{\pi}{2},$$

$$\rho \text{ is positive from } \theta = 0 \text{ to } \theta = \frac{\pi}{2}, \text{ negative from } \theta = \frac{\pi}{2} \text{ to } \theta = \pi,$$

$$\text{positive from } \theta = \pi \text{ to } \theta = \frac{3\pi}{2}, \text{ negative from } \theta = \frac{3\pi}{2} \text{ to } \theta = 2\pi.$$

ρ attains its maximum absolute value for every odd multiple of $\frac{\pi}{4}$.

After θ passes through 2π, the values of ρ repeat. That is, the curve is then complete.

A similar analysis can be made for the curve

(3) $$\rho = a \sin n\theta.$$

It can be shown that if n is an odd number, the curve has n loops, each bearing a family resemblance to the loops of the curve shown. If n is an even number, the curve has $2n$ such loops.

VIII.6 SPIRALS

There are many curves that have a spiral shape. Two of these have considerable interest for us.

The simple algebraic equation,

(1) $$\rho = a\theta,$$

is one of these. It is sometimes known as the Spiral of Archimedes.

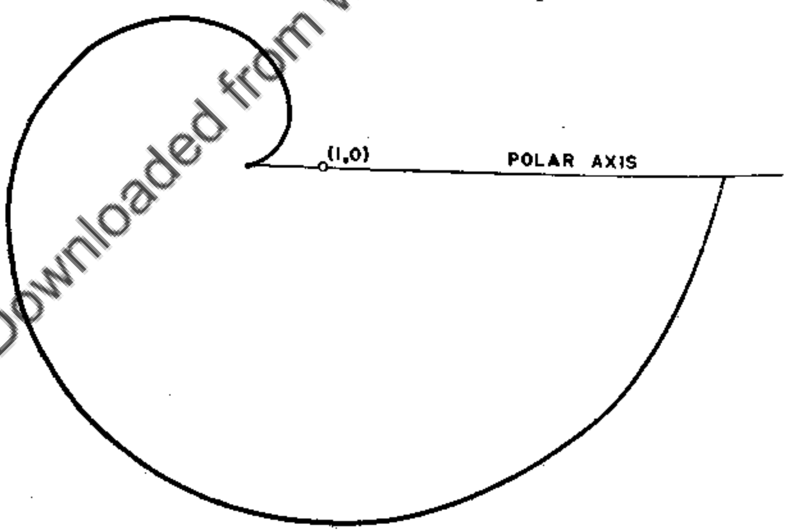

FIG. 73.—SPIRAL OF ARCHIMEDES.

As θ increases by 2π, the radius vector increases by $2\pi a$.

Another spiral curve is given by the equation

(2) $$\rho = e^{a\theta}.$$

It is sometimes known as the logarithmic spiral (Fig. 74). The

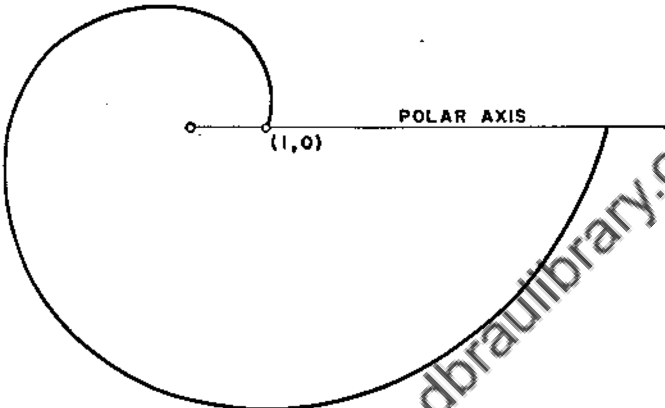

FIG. 74.—LOGARITHMIC SPIRAL.

tangent to the curve at any point makes a constant angle with the radius vector. The proof of this important property is beyond the scope of this work.

VIII.7 THE LEMNISCATE

With the discussion of one more curve, the lemniscate (Fig. 75), we conclude this chapter:

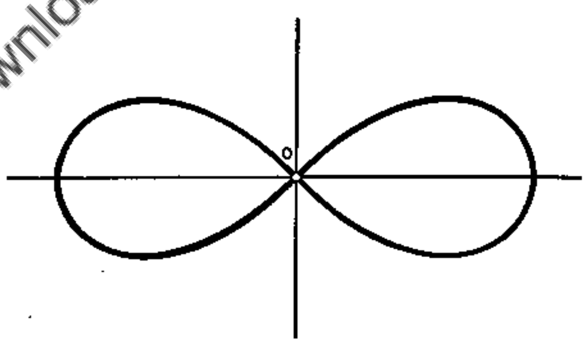

FIG. 75.—LEMNISCATE.

$$(1) \qquad \rho^2 = a^2 \cos 2\theta.$$

Since (1) is not changed when σ is replaced by $-\sigma$ and ρ by $-\rho$ the curve is symmetrical with respect to the polar axis and with respect to the pole. ρ takes on its maximum absolute values for $\theta = 0$, and $\theta = \pi$. ρ is zero when θ is any odd multiple of $\pi/4$. ρ takes on imaginary values whenever $\cos 2\theta$ is negative, i.e., between $\pi/4$ and $3\pi/4$. Thus all the real points will be found when θ ranges from $-\pi/4$ to $+\pi/4$.

EXERCISES

Trace the following curves, naming them, if possible:

1. $\rho = 2 \cos 2\theta$.
2. $\rho = 3 \sin 3\theta$.
3. $\rho = \sin 2\theta$.
4. $\rho = \cos 3\theta$.
5. $\rho = \sin\left(\frac{\theta}{2}\right)$.
6. $\rho = 2 \cos\left(\frac{\theta}{3}\right)$.
7. $\rho = \sin^2 \theta$.
8. $\rho = \sin^2 \theta + \cos^2 \theta$.
9. $\rho^2 = \cos \theta$.
10. $\rho^2 = 4 \sin 2\theta$.
11. $\rho^2 = 9 \cos 2\theta$.
12. $\rho = \frac{\theta}{2}$.
13. $\rho = 2^{\frac{\theta}{\pi}}$.
14. $\rho = \sec \theta + \tan \theta$.
15. $\rho = 2 \tan \theta$.
16. Write the polar equation of the curve $2xy = a^2$.
17. Write the polar equation of the curve $x^2 + y^2 + 4y = 0$.
18. Write the polar equation of the curve $(x^2 + y^2 + ax)^2 = a^2(x^2 + y^2)$.
19. Write the rectangular equation of the curves in Exercises 1, 5, 7, 11, 14.

Find the points of intersection of the following pairs of curves and plot the graphs.

20. $\rho = 4 \cos \theta$, $\rho = 2$.
21. $\rho = a(1 - \cos \theta)$, $\rho = a(1 + \cos \theta)$.
22. $\rho = \sin \theta$, $\rho = 1 - \cos \theta$.
23. $\rho^2 = 5 \cos 2\theta$, $\rho = 2$.
24. $\rho = 2 \cos 2\theta$, $\rho = 2 \cos \theta$.

CHAPTER

IX

GENERAL METHOD OF CURVE TRACING

IX.1 INTRODUCTION

In determining the locus corresponding to a given equation, which is one of the very important phases of analytic geometry, certain general principles offer the advantage of a systematic attack, as well as a reduction in the amount of labor involved. The following pages present some of these general considerations, which have been found helpful in sketching the graphs of equations.

IX.2 SYMMETRY

Two points are said to be symmetrical with respect to a given line if that line is the perpendicular bisector of the line segment

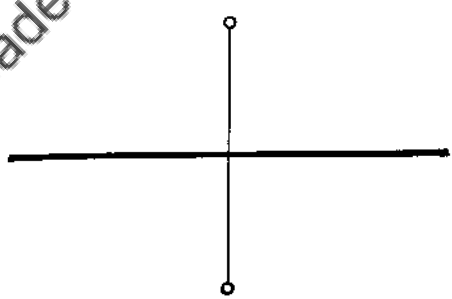

FIG. 76.—SYMMETRY.

joining the points (Fig. 76). Two points are symmetrically placed with respect to the mid-point of the line segment joining them.

It is evident that if we knew in advance that a curve has an axis of symmetry, the labor of drawing the curve would be greatly reduced. There are a few lines such that the symmetry, or lack of symmetry, with respect to them can be easily tested.

1. If y occurs only in even powers, the curve is symmetrical with respect to the x-axis. For, corresponding to any point (x_1, y_1) lying on the curve, there is a second point $(x_1, -y_1)$ whose co-ordinates also satisfy the equation, and these two points are symmetrically placed with respect to the x-axis.

EXAMPLE 1. Test the equation $y^2 = x^3 + x$ for symmetry with respect to the x-axis. (Fig. 77.)

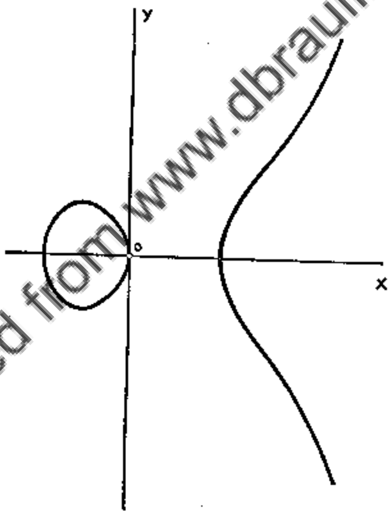

FIG. 77.

There are no odd powers of y, hence for any value of x there are two values of y, numerically equal but with opposite signs, except for $x = 0, 1, \text{ or } -1$, when $y = 0$.

It is easy to see that y is real for

$$-1 \leqq x \leqq 0 \quad \text{and} \quad x \geqq 1,$$

and imaginary for all other values of x. By plotting a few typical points in the range of x for which y is real the curve may be readily drawn.

2. By similar reasoning, if x occurs only in even powers, the curve is symmetrical with respect to the y-axis. The curve $y^2 = x^3 - x$ is not symmetrical with respect to the y-axis, since x occurs in odd powers.

3. If x and y are interchangeable without changing the equation, the curve is symmetrical with respect to the line $y = x$.

If (x_1,y_1) lies on the curve, the point (y_1,x_1) also lies on the curve, and the perpendicular bisector of the line segment joining these two points is $y = x$.

EXAMPLE 2. The curve $xy = 4$ is symmetrical with respect to the line $x - y = 0$, since interchanging x and y leaves the equation the same.

The transformation of rotation of axes, introduced in §VI.5, had for its purpose the reduction of an equation to a form in which axial symmetry would be apparent.

IX.3 CENTRAL SYMMETRY

Another type of symmetry, which is not so easy to recognize in general, is symmetry with respect to a point. If that point is the origin, the test is very simple.

Two points are symmetrically placed with respect to the origin, if the co-ordinates are numerically equal, but opposite in sign, in other words, if the origin is the mid-point of the line segment joining the points.

A curve is symmetrical with respect to the origin, if corresponding to any point, (x_1,y_1), on the curve, there is a second point, $(-x_1,-y_1)$, whose co-ordinates satisfy the equation. The test, whether changing the signs of both x and y leaves the equation unchanged, is made most easily by examining the degree of each term in both x and y. If every term is of even degree (a constant, of zero degree, is classed as of even degree) in x and y together, or, if every term is of odd degree in x and y together, the curve is symmetrical with respect to the origin.

EXAMPLE. The two curves

$$5x^2 - 7xy + 2y^2 = 15,$$
$$13x^3 - 4x^2y + 9y + 4x = 0.$$

are both symmetrical with respect to the origin.

IX.4 INTERCEPTS

By setting $y = 0$ and solving the resulting equation for x, we find the points where the curve crosses the x-axis (the x-intercepts). Similarly the y-intercepts are found by setting $x = 0$ and solving for y.

IX.5 EXCLUDED REGIONS

It may be possible, from an examination of the equation, to determine regions of the plane into which the curve does not go. If so, the drawing of the curve is simplified.

EXAMPLE. Sketch the curve

$$x^3 - x^2y + y + x = 0.$$

The curve is symmetrical with respect to the origin, since every term is of odd degree in x and y.

The curve is not symmetrical with respect to either axis.

The curve crosses the x-axis at the origin, but in no other real points. It crosses the y-axis only at the origin.

Since the equation is linear in y, there is one and only one real point corresponding to any real value of x, except for $x = 1$ and $x = -1$ for which values y is not defined.

If we solve for y,

$$y = \frac{x^3 + x}{x^2 - 1} = \frac{x\,(x^2 + 1)}{(x - 1)\,(x + 1)},$$

we see that when x approaches $+1$ or -1, the numerical value of y increases indefinitely.

When $x > +1$, y is positive.
When $0 < x < +1$, y is negative.
When $-1 < x < 0$, y is positive.
When $x < -1$, y is negative.

We may, then, block off portions of the plane, as shown in Fig. 78, and thus arrive more rapidly at an idea of the curve.

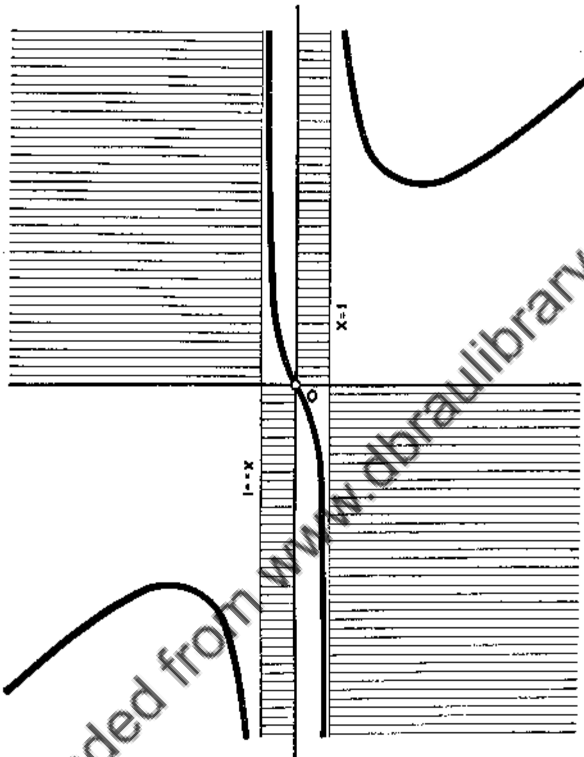

FIG. 78.

IX.6 ASYMPTOTES

In the example of §IX.5, we saw that when x came close to either of the values $+1, -1$, the numerical value of y became increasingly large. If we were to substitute $x = 1$ or $x = -1$, we should find that no value exists for y. The two lines, $x = 1$, $x = -1$, are called **vertical asymptotes** of the curve. An asymptote is sometimes defined as a line that the curve approaches more and more closely as it recedes indefinitely from the origin.

In a similar way, we sometimes find **horizontal asymptotes,** lines such as $y = c$. For this case no finite value of x exists corresponding to $y = c$.

EXAMPLE. Sketch the curve $xy + 3x - 2y = 0$.

Solving for y, we note that $y = -3x/(x - 2)$. Hence, y is negative for values of x greater than 2, positive for all other positive values of x, and negative for all negative values of x. The line $x = 2$ is a vertical asymptote.

If we solve for x, we find

$$x = \frac{2y}{y + 3}.$$

Hence, the line $y + 3 = 0$ is a horizontal asymptote. x is positive for all positive values of y, negative for $-3 < y < 0$, and positive for all other values of y. The curve (Fig. 79) passes through the origin.

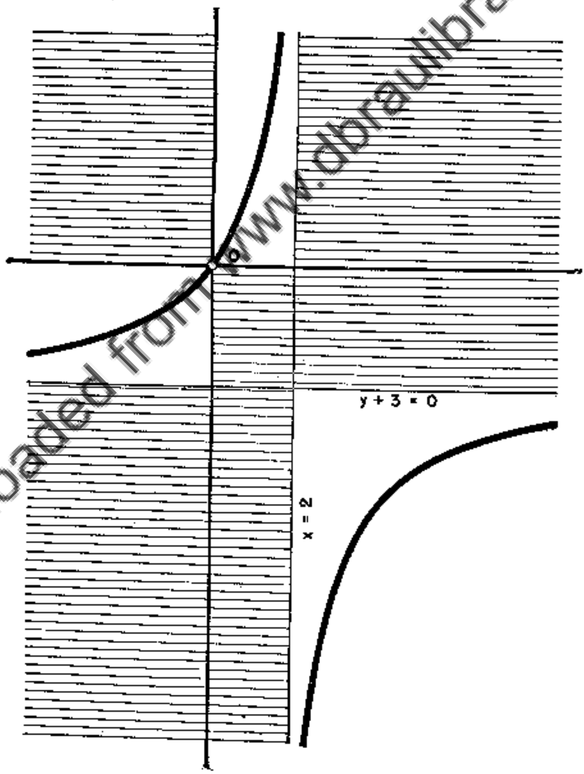

FIG. 79.

Of the many curves of degree three, there are a few of such importance that we should be familiar with them. The principles given in the preceding pages are easily applied in drawing the curves of the following exercises.

EXERCISES

Sketch the following curves.

1. The cubical parabola, $y = x^3$.
2. The semicubical parabola, $y^2 = x^3$.
3. The cissoid, $y^2(a - x) = x^3$.
4. The strophoid, $y^2(a - x) = x^3 + ax^2$.
5. The folium, $x^3 + y^3 = 3xy$.
6. The parabola, $x^{1/2} + y^{1/2} = a^{1/2}$.
7. The witch, $y = \dfrac{8a^3}{x^2 + 4a^2}$.
8. $y = x^3 - ax^2$.
9. $y = x^3 - 3x^2 + 2x$.
10. $y(a^2 - x^2) = x^3 - 4a^2x$.
11. $y^3 - y^2x + x + y = 0$.
12. $xy + 3y - 2x = 0$.
13. $y^2x - x^3 + 2x^2 - x + 2 = 0$.
14. $y^2 = x^3 - ax^2$.
15. $y^2 = x^3 + ax^2$.
16. $y^2 = x^3 - x$.
17. $y^2 = x^3 + x$.

IX.7 TRIGONOMETRIC FUNCTIONS

We recall from trigonometry that if we construct a circle with radius r, with center at the origin and draw any angle $\theta = AOP$,

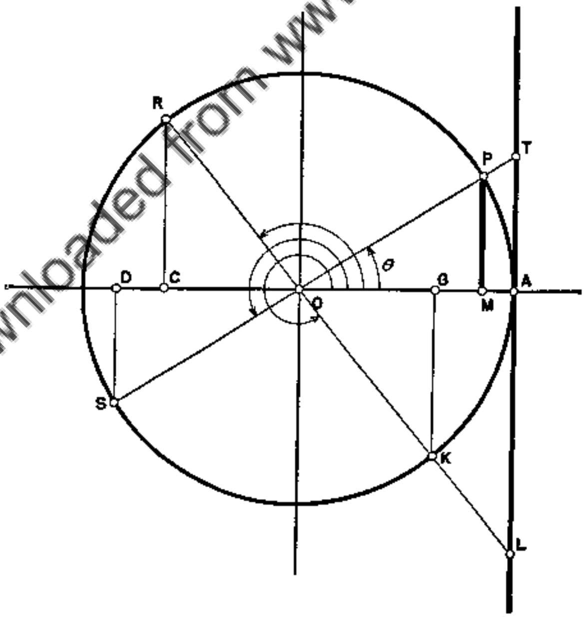

FIG. 80.

as in Fig. 80, the length of the perpendicular $MP = r \sin \theta$. If OP meets the tangent to the circle at A in the point T, the length $AT = r \tan \theta$, and $OT = r \sec \theta$.

If θ is an acute angle, all three of these values are positive. When θ lies between $\pi/2$ and $3\pi/2$, illustrated in Fig. 80 by angles AOR, AOS, the secant is negative, since the new position of point T is in the opposite direction from the terminal line of the angle. For angle AOR, OR would be considered positive, and hence OL would be negative.

If θ were an obtuse angle, the segment AT representing $\tan \theta$ (AL in the figure) would be negative, while for an angle in the third quadrant, such as AOS, the value of the tangent is again positive.

To graph the equation

(1) $$y = \sin x,$$

note that the angle is expressed in radian measure. ($180° = \pi$ radians.) We can, of course, substitute values of x, look up the corresponding values of y in a table, and plot points. It is easier, however, to apply a graphic method.

Divide a circle of unit radius into any number of equal parts. (32 parts give a reasonable degree of accuracy.) Starting from the origin, lay off a succession of equal segments each equal to $\pi/16$. (If we use for each of these segments the chord corresponding to the angle $\pi/16$, the result, while not accurate, involves an error of approximately 6/10 of 1%.) Draw vertical lines through these division points, and lay off on these the ordinates equal to the corresponding values of MP taken from the circle.

It is a saving of time and effort to draw the unit circle with center on the x-axis and run horizontal lines through the division points on the circle, marking the intersections with the corresponding vertical lines. (See Fig. 81.)

A similar process may be used to sketch the curve

$$y = \tan x.$$

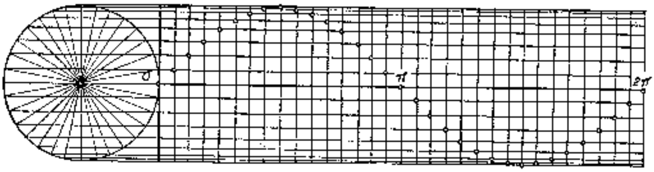

FIG. 81.—y = sin x.

If we remember that $\cos x = \sin (\pi/2 - x)$, we can easily devise a method for sketching the curve

$$y = \cos x.$$

These curves are called **periodic** curves, since the same values of y are repeated at regular intervals in the horizontal direction. If we multiply x by any constant, k, the effect is to change the period. Thus the curve $y = \sin 2x$ has a period half as long as that of the curve $y = \sin x$.

IX.8 EXPONENTIAL AND LOGARITHMIC CURVES

The curves

(1)
$$y = e^{x}$$
$$y = e^{-x}$$

can be drawn with the help of the table of exponentials. (Fig. 82.)

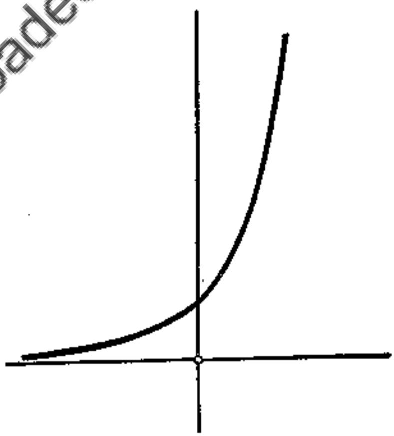

FIG. 82.—y = eˣ.

We note that y is never negative, that for negative values of x, y is between 0 and 1, and decreases toward zero as x takes on large negative values, that the curve crosses the y-axis at (0,1), and that y increases very rapidly as x takes on increasing positive values.

Similarly, a table of logarithms enables us to plot the curve,

$$y = \log_e x. \tag{2}$$

It is worth noting that this curve and the curve $y = e^x$ differ only in the fact that the interchange of x and y causes an interchange of the two curves since (2) may be written as $x = e^y$.

The exponential functions are very important in many applied problems dealing with such matters as growth, rates of chemical reactions, radioactivity, etc.

EXERCISES

Draw the following curves:

1. $y = \sin x$.
2. $y = \cos x$.
3. $y = \tan x$.
4. $y = \cot x$.
5. $y = \sec x$.
6. $y = \csc x$.
7. $y = \sin 2x$.
8. $y = 2 \cos 2x$.
9. $y = \sin x + 2 \cos x$.
10. $y = \tan 2x$.
11. $y = \cos\left(\frac{x}{2}\right)$.

With the help of tables of exponentials, draw the following curves:

12. $y = e^x$.
13. $y = e^{-x}$.
14. $y = e^{\frac{x}{2}}$.
15. $y = -e^{-\frac{x}{2}}$.
16. Plot the catenary, $y = a/2(e^{x/a} + e^{-x/a})$, for $a = 2$.
17. Plot the probability curve, $y = e^{-x^2}$.

By means of a table of logarithms, draw the following curves:

18. $y = \log_{10} x$ and $y = \log_e x$.
19. $y = \log_e x^2$.
20. $y = \log_{10}(x + 1)$.
21. $y^2 = \log_{10} x$.

GENERAL METHODS OF CURVE TRACING

Symmetry

with respect to x-axis	y only in even powers
with respect to y-axis	x only in even powers
with respect to line $y = x$	x and y interchangeable
with respect to origin	changing signs of both x and y does not change the equation

Intercepts

set $y = 0$ and solve for x
set $x = 0$ and solve for y

Extent of curve

Block off regions into which curve does not go

Asymptotes

Vertical—solve for y and set denominator = 0
Horizontal—solve for x and set denominator = 0

Plot points whose co-ordinates satisfy the equation

CHAPTER

X

PARAMETRIC EQUATIONS

X.1 INTRODUCTION

There are times when a direct relationship between two variables, x and y, is difficult to express by a single equation. Or even if such an equation can be found, it is so involved as to be difficult, if not impossible, to determine corresponding values. For example, the equation of the cycloid, the curve traced out by a point on the rim of a wheel rolling along a straight line, is

$$x = a \operatorname{arc} \cos \left(\frac{a - y}{a} \right) - \sqrt{2ay - y^2}.$$

If we had to find values of y corresponding to values of x, we should find that the problem presents some practical difficulties.

But if we express x and y in terms of the angle through which the wheel has turned from an initial position, we have

$$x = a(\theta - \sin \theta),$$
$$y = a(1 - \cos \theta).$$

Then the determination of the co-ordinates of any point on the curve becomes a matter of substituting values of θ.

When x and y are both expressed as functions of a third, independent variable, we may think of them as being determined by, or measured in terms of this independent variable. We speak of this third or independent variable as a **parameter.** Then the equations are known as **parametric equations.**

The parametric equations often arise from a consideration of the conditions under which motion takes place, as in the case of a projectile, assumed to be acted upon only by a constant gravitational attraction of the earth. Here the equations are,

$$y = v_0 \sin \alpha \cdot t - \frac{1}{2} gt^2,$$
$$x = v_0 \cos \alpha \cdot t,$$

where v_0 is the speed at the start, when $t = 0$, α is the angle that the path makes with the horizontal at the start, and t is the time in seconds.

In graphing such equations, we can, of course, substitute values for the parameter, compute the corresponding values of x and y, and plot in the usual manner. However, except for integral values of the parameter, the computation is apt to prove cumbersome, and the integral values will ordinarily give points so widely separated that it is difficult to see from them what the curve is like.

To overcome this difficulty, it may be found convenient to have fairly accurate drawings of a few standard curves, such as $s = t^2$, $s = t^3$, from which values can be found graphically for any value of t, and thus the co-ordinates of points on the curve can be found without a lot of laborious arithmetic.

EXAMPLE. Sketch the curve

$$x = t^2 + 2t,$$
$$y = t^3 - t^2 + 1.$$

Using the graphs of t^2 and t^3 in Fig. 83, we can readily find the values

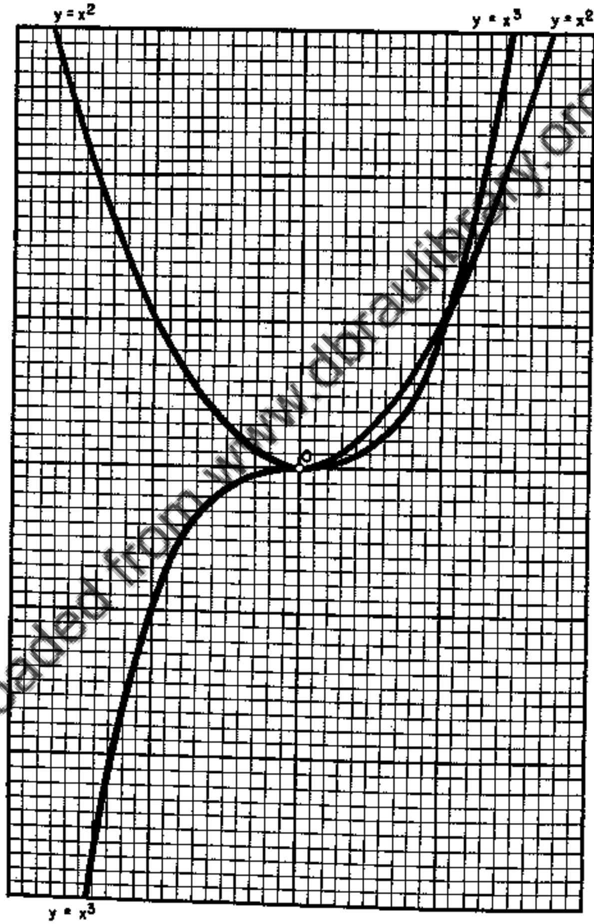

FIG. 83.—$y = x^2$, $y = x^3$.

of x and y that correspond to a convenient value of t. By plotting the points we can draw the graph, as shown in Fig. 84.

A few important points, such as the intercepts on the axes may also be readily found by direct methods.

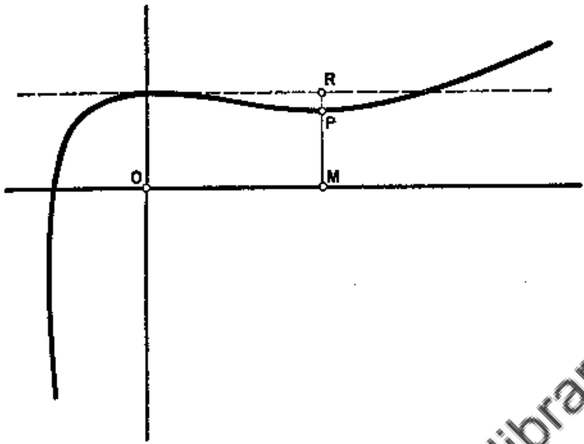

FIG. 84.—$x = t^2 + 2t$, $y = t^3 - t^2 + 1$.

X.2 CONSTRUCTION OF t^n

Construct two lines at right angles, OA, OB; take the distance $OA = 1$. On the line OB take $OB = t$, any arbitrary length. Construct BC perpendicular to AB. Then, from plane geometry,

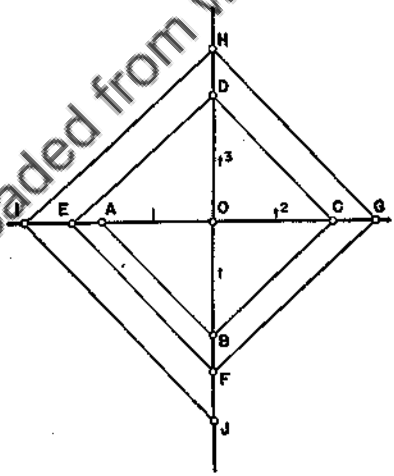

FIG. 85.—GRAPH OF t^n.

$OC = t^2$. Continuing to draw perpendiculars, as shown in Fig. 85, we find $OD = t^3$, $OE = t^4$, $OF = t^5$, etc. If OA were taken $= a$, we should find $OC = t^2/a$, $OD = t^3/a^2$, etc.

EXAMPLE 1. Sketch the curve

$$x = 1 + t,$$
$$y = 1 - 2t^2.$$

Draw through A (1,1) vertical and horizontal lines (Fig. 86). On the

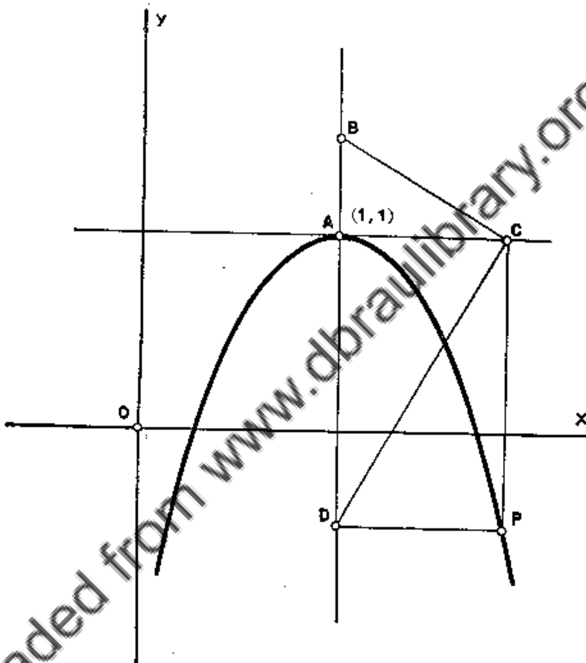

FIG. 86.—EXAMPLE 1.

vertical line locate B, ½ unit above A. Choose an arbitrary point C on the horizontal line. Let $AC = t$. The vertical line through C is the line $x = 1 + t$. Draw $BCD = 90°$. $BA \cdot AD = t^2$.

Then

$$AD = \frac{t^2}{BA} = \frac{t^2}{(\frac{1}{2})} = 2t^2.$$

The horizontal line through D is

$$y = 1 - 2t^2.$$

Then P is a point of the required curve.

EXAMPLE 2. Sketch the semicubical parabola $x = t^2$, $y = t^3$.

A direct application of the method suggested in the discussion leads to points of the curve. A is the point $(-1,0)$, B is any point on the

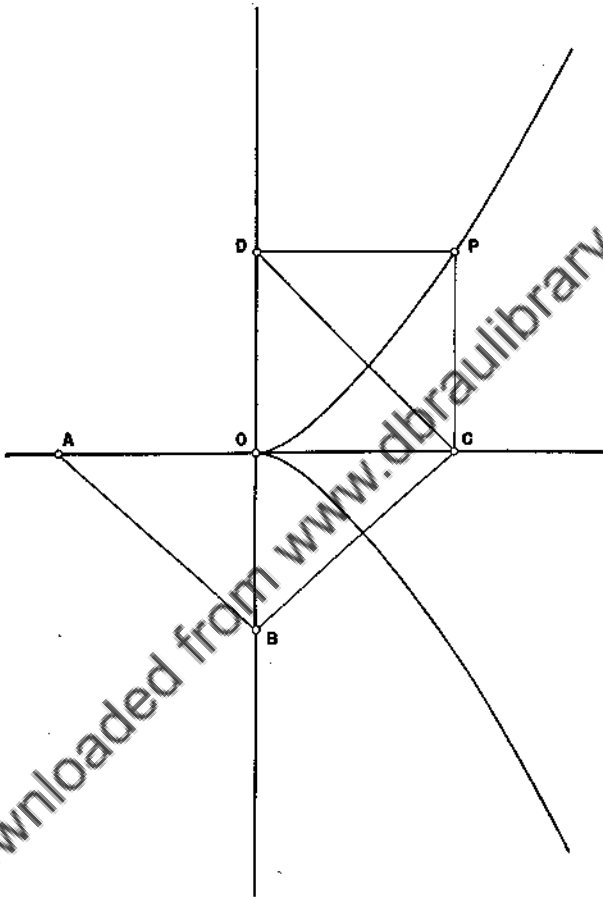

FIG. 87.—EXAMPLE 2: SEMICUBICAL PARABOLA.

y-axis with $OB = t$, BC is perpendicular to AB. (Fig. 87.)

The vertical line CP has the equation

$$x = t^2.$$

CD is perpendicular to BC. The horizontal line DP has the equation

$$y = t^3.$$

Therefore P is a point of the curve.

EXERCISES

Plot or construct the curves given by the following parametric equations either by calculating values for x and y, or by using graphic methods:

1. $x = t, y = t^3.$
2. $x = t^2, y = t - 1.$
3. $x = t^3, y = t^2.$
4. $x = \left(\dfrac{2t-1}{t+1}\right), y = (t+1)\sqrt{t-1}.$
5. $x = t^3 - 5t + 6, y = 3t^2 - 5t + 2.$
6. $x = t^3 + t, y = t^3 - t.$
7. By eliminating t, obtain the **rectangular** or **Cartesian** equations of the curves in Exercises 1–6.

   Ans. 1. $y = x^3$. **2.** $y^2 - x + 2y + 1 = 0$. **4.** $2xy - y - 3 = 0$.
8. $x = \dfrac{a(1-t^2)}{1+t^2},$

   $y = \dfrac{at(1-t^2)}{1+t^2}.$ (Strophoid.)
9. $x = \dfrac{3at}{1+t^3},$

   $y = \dfrac{3at^2}{1+t^3}.$ (Folium.)
10. By substituting $y = tx$ in the equation $y^2 = ax$ show that

$$x = \frac{a}{t^2}, \quad y = \frac{a}{t}$$

are parametric equations of an ellipse.

X.3 TRANSCENDENTAL PARAMETRIC EQUATIONS

If the co-ordinates x and y are expressed in terms of the trigonometric functions of a parameter, our knowledge of these functions and their relationship may suggest methods of construction.

Consider the curve,

$$x = a \cos \theta,$$
$$y = b \sin \theta.$$

A suggested construction is as follows: Draw two concentric circles with radii a,b and center at the origin. Draw a line OA, making an angle θ with the x-axis (see Fig. 88) and cutting the first circle in A, the second in B. Through A draw a vertical line MA, whose equation is

$$x = a \cos \theta.$$

Through B draw a horizontal line, whose equation is

$$y = b \sin \theta.$$

These two lines meet in the point P, which is a point of the desired curve.

If we eliminate θ, by making use of the identity

$$\cos^2 \theta + \sin^2 \theta = 1,$$

we see that the curve is an ellipse with the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is probably the easiest method of constructing points on the ellipse.

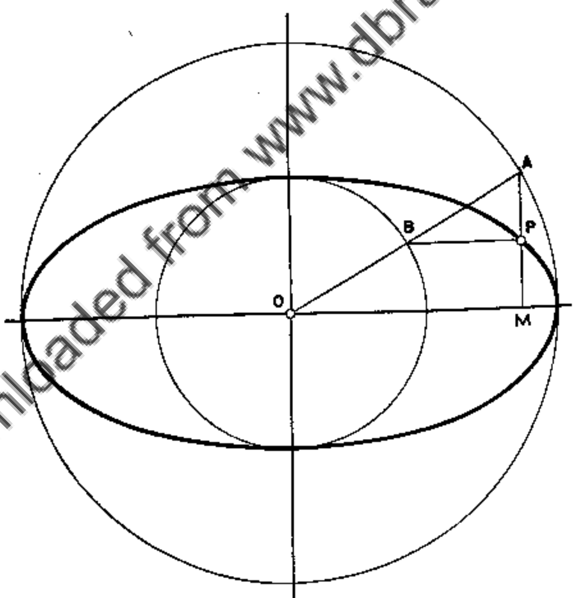

FIG. 88.—CONSTRUCTION OF ELLIPSE.

EXAMPLE. Sketch points of the curve

$$x = a \sec \theta,$$
$$y = b \tan \theta.$$

Fig. 89 illustrates the construction of two points of the curve, P, P'.

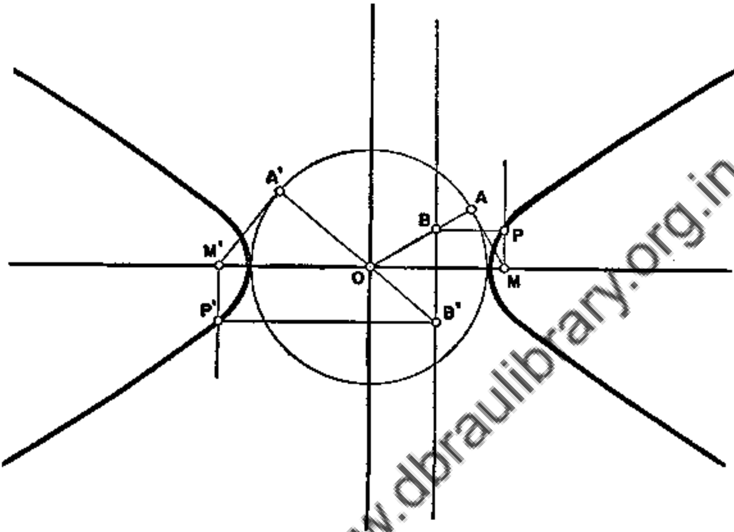

FIG. 89—HYPERBOLA.

P corresponds to an acute angle θ, P' to an angle in the second quadrant. The circle is of radius a, while the line BB' is at a distance b to the right of the origin.

AM is tangent to the circle at A. B is the point in which OA intersects the line BB'. BP is horizontal and has the equation

$$y = b \tan \theta.$$

MP is vertical and has the equation

$$x = a \sec \theta.$$

Then P is a point of the curve.

The parameter θ may be eliminated by the use of the identity

$$\sec^2 \theta - \tan^2 \theta = 1,$$

giving the equation of the curve,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

EXERCISES

1. Carry through the details of eliminating the parameter θ in both of the illustrative examples.
2. By the suggested method construct a number of points on each of the following curves, and sketch the curves:

(a) $x = 3 \cos \theta$, $y = 2 \sin \theta$.

(b) $x = \cos \theta$, $y = 4 \sin \theta$.

(c) $x = 4 \sec \theta$, $y = 3 \tan \theta$.

(d) $x = 3 \sec \theta$, $y = 4 \tan \theta$.

(e) $x = a (\cos t + t \sin t)$, $y = a (\sin t - t \cos t)$.

(f) $x = a (2 \cos t - \cos 2t)$, $y = a (2 \sin t - \sin 2t)$.

3. Devise a method for constructing points of each of the following:

(a) $x = 4 \sec \theta$, $y = 4 \cos \theta$.

(b) $x = 3 \cot \theta$, $y = 3 \tan \theta$.

(c) $x = 1 + \tan \theta$, $y = \tan \theta + \sec \theta$.

(d) $x = a(\theta - \sin \theta)$, $y = a (1 - \cos \theta)$. (The cycloid.)

(e) $x = a\theta - b \sin \theta$, $y = a - b \cos \theta$. $a < b$, prolate cycloid, $a > b$, curtate cycloid.

(f) $x = a + b \cos \theta$, $y = c + d \sin \theta$.

4. Determine the rectangular equation which corresponds to 3(*f*) and show that it is the equation of a circle.

X.4 THE HYPOCYCLOID OF FOUR CUSPS

This curve, which is of considerable interest, is given by the equations

$$(1) \qquad x = a \cos^3 \theta, \quad y = a \sin^3 \theta.$$

These equations suggest the following construction method. Draw a circle of radius a (Fig. 90). Draw any radius OA, making an angle θ with the x-axis. Draw AM perpendicular to OM, MR perpendicular to OA, RS perpendicular to OM. $OS = OR \cos \theta = OM \cos^2 \theta = OA \cos^3 \theta$. Draw AN perpendicular to ON, NQ perpendicular to OA, LQ perpendicular to ON.

$$OL = OQ \sin \theta = ON \sin^2 \theta = OA \sin^3 \theta.$$

Then LQ meets RS in P, the desired point on the curve.

We note that the entire curve lies inside the circle and that the curve has four points on the circle.

If we eliminate the parameter θ, we find the rectangular equation

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

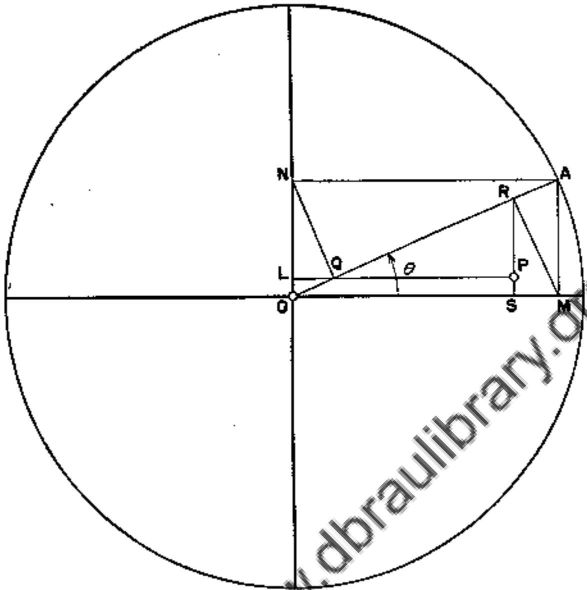

FIG. 90.—HYPOCYCLOID.

EXERCISES

1. Show the steps in finding the rectangular equation of the hypocycloid of four cusps.

2. Sketch the curve
$x = 4\cos^3\theta,$
$y = 4\sin^3\theta.$

3. Sketch the curve
$x = 5\cos^3\theta,$
$y = 3\sin^3\theta.$

4. Devise a construction for the curve
$x = 4\cos^3\theta,$
$y = 2\sin\theta.$

5. Find the rectangular equations for the curves in Exercises 3 and 4.

6. Sketch the curve
$x = 2\cos\theta - \cos 2\theta,$
$y = 2\sin\theta - \sin 2\theta.$

7. Sketch the curve
$x = 3\cos\theta - \cos 3\theta,$
$y = 3\sin\theta - \sin 3\theta.$

8. Sketch the curve
$x = 3\cos\theta - 2\cos 3\theta,$
$y = 3\sin\theta - 2\sin 3\theta.$

9. Sketch the curve
$x = 2\cos\theta + \cos 2\theta,$
$y = 2\sin\theta - \sin 2\theta.$

10. Sketch the curve
$$x = (a - b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right),$$
$$y = (a - b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right).$$

11. Sketch the curve
$$x = (a + b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right),$$
$$y = (a + b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right).$$

CHAPTER

XI

EMPIRICAL EQUATIONS

XI.1 INTRODUCTION

In the development of formulas and equations of loci in the preceding chapters, the geometric conditions involved have been very specific and of such a nature as to lend themselves more or less readily to algebraic formulation. In other words, we have known that there existed a definite functional relationship between the variables, and building an equation has been merely the process of translating known relationships into algebraic language.

There are a great many cases in which, although we are convinced that a relationship exists, we do not know the precise nature of that relationship, or may be unable to express that relationship in mathematical language. For example, at any given time a certain thermometer gives a definite temperature reading. This implies that there is a definite relationship between temperature and time. But that relationship is so involved that the only way we can, at present, exhibit it is by means of a chart or a table of corresponding values.

The fundamental assumption on which all science is based is that **all events happen according to some law.** The search for relationship of cause and effect that will enable one to predict what will happen under given circumstances would be utterly futile and meaningless if events were governed purely by chance.

For the scientist, the existence of a law is evident from the fact that his observations are grouped in such a way as to indicate, as the statisticians say, a **significant trend.**

Once he has observed such a significant trend, his next problem is the formulation of an equation between the variables he has observed, which will enable him to predict with some degree of precision what will happen under circumstances of like nature.

The systematic attack on this problem of formulating equations to express relationships observed by experiment constitutes the present chapter on **empirical equations.** In general, we cannot hope for absolute precision in our results, for two reasons. First, not knowing all the factors involved, we cannot be sure that the type of equation used expresses the actual relationship. Second, there may be, and probably are, errors in the experimental observations. We shall, however, expect that our equation shall approximate the observed relationships as closely as possible, while retaining, for economy of time and effort, a simple form.

The determination of empirical equations is a twofold problem, which attempts to answer the questions,

(a) What type of equation most nearly expresses the true relation between the variables?

(b) What are the best values of the arbitrary constants or parameters that occur in the chosen type of equation?

For example, if we desire an equation that will express the weekly average temperature as a function of the time in weeks, we might assume, since the fluctuations are apparently periodic, with a period of approximately 52 weeks, that

$$u = a \cos \frac{\pi}{26} (t + \alpha),$$

where u is the weekly average temperature, t the time in weeks. We then undertake to determine the arbitrary constants, a and α, in such a way that the values of u computed from the equation may correspond as closely as possible with the available data.

XI.2 THE SCATTER DIAGRAM

We shall limit our discussion of empirical equations to two variables, even though many of the principles developed can be extended to more than two variables. For convenience in discussion, we may suppose that x and y are the variables to be related, although in practice we work with any convenient symbols as variables.

Suppose the observed values of x are $x_1, x_2, x_3, \ldots$, and the corresponding values of y are $y_1, y_2, y_3 \ldots$. Each pair of corresponding values (x_i, y_i) may be represented graphically by a point in the xy-plane. The geometric figure formed by the entire group of points determined in this way is called a **scatter diagram.**

If there is a significant relationship between x and y, the points of the scatter diagram will tend to cluster around some curve. It is the equation of this curve which we wish to determine.

EXERCISES

1. The number of degrees granted by a certain college each year for a ten-year period was as follows:

Year, x:	1	2	3	4	5	6	7	8	9	10
Number of degrees, y:	128	131	145	163	228	243	300	322	370	430.

Construct a scatter diagram from the given data and draw a smooth curve which seems to show the trend.

2. Some experimental observations of the temperature t and the corresponding pressure p, in millimeters, of superheated steam give the following data:

t (Centigrade):	0	10	20	30	40	50	60	70	80	90	100
p (Millimeters):	4.6	9.2	17.4	31.6	55.0	92.2	149.2	233.8	355.5	526.0	760.0.

Construct the scatter diagram, and sketch a smooth curve to represent the relationship.

3. A group of students received the following grades in algebra and trigonometry:

Algebra:	A	A	B	B	C	A	D	F	D	F	C	B	A	C
Trigonometry:	A	B	C	A	C	F	C	B	D	F	B	C	D	D.

Assuming $A = 1$, $B = 2$, $C = 3$, $D = 4$, $F = 5$, plot the scatter diagram, and draw a straight line which seems to indicate the trend.

4. Construct a scatter diagram, and draw a straight line which approximately fits the following values of x and y:

x:	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y:	10	19	29	46	51	66	78	89	101	113.

Assume the equation of the straight line to be in the form

$$y = mx + b$$

and determine the slope, m, and the y-intercept, b, from the graph. Using the equation obtained in this manner, compute the value of y for each of the given values of x and compare these values with the values given in the table.

5. Construct a scatter diagram from the data in Exercise 1, using x as abscissa, and log y as ordinate. Note that the points constructed in this manner tend to lie on a straight line. Draw the line.

XI.3 TYPES OF EMPIRICAL EQUATIONS

The determination of the particular curve that will adequately represent the relationship between the variables is a matter calling for the application of skill and some intelligent guess-work. It is important, from the standpoints of both determination of the curve and its application, that the form be as simple as possible.

The way in which the points of the scatter diagram are clustered forms a rough guide for a tentative sketch of a smooth curve, which can be compared with curves whose equations are known, in order to select the proper type.

A knowledge of the phenomena giving rise to the data often forms an important clue. Thus the existence of vertical asymptotes (where y becomes infinite), of maximum or minimum values, of periodic characteristics, would be valuable aids in choosing the type of curve.

An extensive discussion of the types of curves that might occur would be beyond the scope of this text. A few, which are fairly common and easily recognized, will suffice to care for most of the cases usually encountered. The five that we shall consider are:

(a) The straight line, $y = mx + b$.
(b) The parabola, $y = a + bx + cx^2$.
(c) The exponential curve, $y = ab^x$.
(d) The power curve, $y = ax^k$.
(e) The hyperbolic curve, $y = \dfrac{a}{b + x}$.

EXERCISES

1. Graph on the same axes the curves $y = ax^n$ for a fixed value of a, and $n = 1, 2, 3, 4$, and 5.
2. If $z = \log x$ and $w = \log y$, show that the equation $y = ax^n$ leads to a linear equation in z and w.
3. Show that if $\log x$ and $\log y$ are connected by a linear equation of the form

$$\log y = b + c \log x,$$

y and x are related by an equation of the form $y = ax^c$.

4. Show that if the values of y corresponding to the values $x = 1, 2, 3, \ldots$ form an arithmetic progression, the relation between x and y may be written

$$y = mx + b,$$

where m and b are constants.

5. Show that if $y = ab^x$, the values of y corresponding to $x = 1, 2, 3, \ldots$ form a geometric progression.

6. Show that if the values of $\log y$ corresponding to $x = 1, 2, 3, \ldots$ form an arithmetic progression, the values of y form a geometric progression, and that the relation between x and y may be written

$$y = ab^x.$$

7. If the values of y corresponding to $x = 1, 2, 3, 4, \ldots$ are $y_1, y_2, y_3, y_4, \ldots$, and we let $z_1 = y_2 - y_1$, $z_2 = y_3 - y_2$, $z_3 = y_4 - y_3, \ldots$, show that when $y = a + bx + cx^2$, the values $z_1, z_2, z_3, \ldots$ form an arithmetic progression.

8. Discuss the essential character of the curves defined by

$$y = \frac{a}{b - x},$$

and construct graphs for different values of b, keeping a fixed.

9. Show that Boyle's law $pv = c$, where p is pressure and v is the volume of a gas under constant temperature, is a special case of the hyperbolic equation

$$y = \frac{a}{b + x}.$$

10. Plot the curves defined by $y = ae^{-k^2x^2}$, keeping a fixed and letting $k = 0.5, 0.25, 0.1, 2$. Could this form of an equation be used when y takes on both positive and negative values? When y approaches zero as x increases? When y is zero for $x = 0$?

11. Could the results of Exercises 3, 4, 6, 7 help in selecting type curves? If so, formulate a guiding principle in each case, in your own words.

XI.4 DETERMINATION OF THE ARBITRARY CONSTANTS

In general, the type forms of empirical equations involve arbitrary constants that must be determined from the corresponding values of the variables found by observation. Thus, if the data call for a straight line, $y = mx + b$, we should be able to determine the constants, m and b, from the experimental data, by some logical method.

If we knew the **exact** values of y corresponding to two distinct values of x, we could determine the line, since a straight line is determined by any two of its points.

In general, if we knew the **exact** values of y for as many values of x as there are constants to be determined, we could substitute and find these constants.

The trouble is that we do not **know**, in experimental data, that our observations are exact. In fact, since they are observations, we know that they may be subject to error. Therefore, instead of finding the **one** curve, we must find a curve that we judge to be the best fit for all the observations, whose number will normally be considerably greater than the number of constants to be determined.

A rather direct method would be to make the scatter diagram and draw the smooth curve that seems to fit best. Then, noting the co-ordinates of several points on this curve, one could compute the values of the necessary constants. Special ruled papers have been devised to assist in this sort of work, but, at best, the method is open to the serious criticism that it rests **on the personal judgment of the individual,** and hence may lead to different interpretations of the same data.

Aside from the purely graphical method, there are three well-established methods. They are

(a) the method of averages,
(b) the method of moments,
(c) the method of least squares.

The last of these three we shall omit, since its development requires a knowledge of more advanced mathematics. This omission will not seriously handicap us, since the application of least squares to fitting a straight line or a parabola leads to exactly the same computation as is found by the method of moments.

XI.5 METHOD OF AVERAGES

The method of averages is most easily presented by means of a special example.

EXAMPLE. Determine the empirical equation fitting the following corresponding values of x and y:

x:	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
y:	10	19	29	46	51	66	78	89	101	113.

The scatter diagram corresponding to this table of values is shown in Fig. 91. The graph shows we may expect a good fit by using a straight

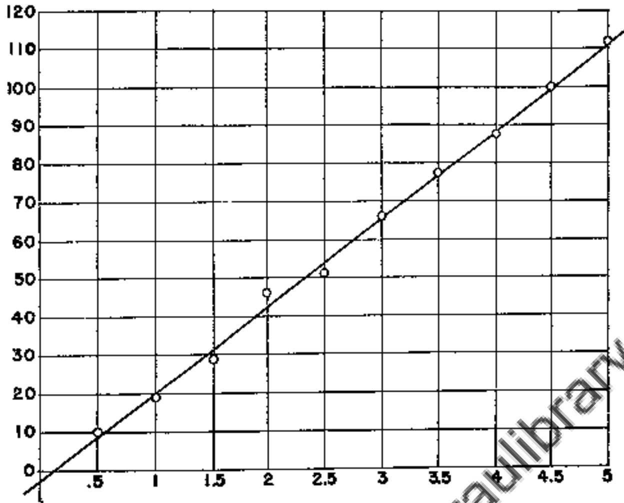

FIG. 91.

line, whose equation we assume to be

$$y = mx + b.$$

Let us divide the ten pairs of values into two groups (the number of groups equal to the number of constants to be determined). Suppose the first group consists of the first five pairs of values. There is a point whose x is the arithmetic mean, or average, of the x's of these five pairs, and whose y is the arithmetic mean of the y's. Similarly the second group of five pairs of values leads to a second point. These two ***mean*** or ***average*** points determine a straight line, which we assume to be the desired straight line fitting the data.

In the particular case, for the first mean point,

$$x = \left(\frac{0.5 + 1 + 1.5 + 2 + 2.5}{5}\right) = 1.5,$$

$$y = \left(\frac{10 + 19 + 29 + 46 + 51}{5}\right) = 31.$$

The second such point is

$$x = \left(\frac{3 + 3.5 + 4 + 4.5 + 5}{5}\right) = 4,$$

$$y = \left(\frac{66 + 78 + 89 + 101 + 113}{5}\right) = 89.4.$$

Substituting these co-ordinates in the equation $y = mx + b$, we have the equations,

$$31 = 1.5m + b,$$
$$89.4 = 4m + b.$$

Solving these two equations, we find $m = 23.36$, $b = -4.04$, and the equation of the straight line is

$$y = 23.36x - 4.04.$$

It should be recognized that the result is dependent upon the manner in which the two groups are set up. For example, if we had taken the first, third, fifth, seventh and ninth pairs of values for the first group, and the others for the second, the mean points would have been (2.5, 53.8) and (3, 66.6). The corresponding straight line would be

$$y = 25.6x - 10.2.$$

Since a slight variation in the position of the mean points has less influence when the points are well separated than when they are close together, it is wise to select the groups with this in mind.

The extension of the method of averages to equations with more than two constants is fairly obvious. It involves forming a number of groups corresponding to the number of constants to be determined. Practically, the method is limited to equations that are linear in the constants, like those in which y is a polynomial function of x, or which can be reduced to linear equations in the constants, as, for example, by taking logarithms of one or both members of the equation.

EXERCISES

1. Determine an equation of the form $y = mx + b$ from the following data:

x:	1	2	3	4	5	6
y:	5.8	5.6	5.0	4.8	4.5	4.1.

2. Find a and b from the following data, under the assumption that $y = ax^2 + b$:

x:	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y:	10	18	30	47	70	97	129	165.

3. Observed values of P, the power developed at one end of a transmission line and W, the work done at the other end, are

P:	50	100	150	200	250	300
W:	49	92	135	175	210	230.

If we assume $W = aP - bP^3$, determine values for a and b by the method of averages.

4. The melting point of a zinc and lead alloy is given by

$$t = az^2 + bz + c$$

where t is the temperature and z is the percentage of zinc. Determine the constants a, b, and c, from the following data:

z:	60	50	40	30	20	10
t:	186	205	226	250	276	305.

5. The number of bacteria in a given culture at various times is assumed to follow a law $N = a \cdot 10^{bt}$.

Observations yield the following data:

t:	0	1	2	3	4	5
N:	150	271	501	914	1665	3000.

Construct a scatter diagram using t and $\log N$ as co-ordinates. Assuming that these lie on a straight line, determine a and b from the equation

$$\log N = \log a + bt.$$

6. Given the following data:

x:	1	2	3	4	5	6
y:	4.0	7.1	10.4	13.4	16.6	19.6

construct a scatter diagram using $\log x$ and $\log y$ as co-ordinates. Assume that these points tend to lie on a straight line $\log y = A + B \log x$ and that as a result $y = ax^b$. Determine a and b.

XI.6 METHOD OF MOMENTS

Let $x_1, x_2, x_3, \ldots, x_n$ be a set of values of an independent variable, x, and $y_1, y_2, y_3, \ldots, y_n$, the corresponding values of a variable, y, which depends on x. The quantity

(1) $$x_1{}^k \cdot y_1 + x_2{}^k \cdot y_2 + x_3{}^k \cdot y_3 + \ldots + x_n{}^k \cdot y_n = M_k(y),$$

where k is zero or a positive integer, is called the **kth moment** of the values of y, relative to the origin of x. If we use the notation

(2) $$\sum_{i=1}^{n} u_i = u_1 + u_2 + u_3 + \ldots + u_n$$

(to be read **the sum as to** i **from one to n of u sub** i, and to be interpreted as **the sum of all possible terms of the form** u_i **when** i **takes on successively all integral values from 1 to n, inclusive**), we may write the expression for the kth moment as

(3) $$\sum_{i=1}^{n} x_i{}^k y_i.$$

If there is no possibility of confusion as to meaning, we may abbreviate by writing $\Sigma\, x^k y$.

(4) Thus the zeroth moment is $\sum_{i=1}^{n} (x_i{}^0 y_i) = \sum_{i=1}^{n} y_i = \Sigma y$,

the first moment is $\sum_{i=1}^{n} (x_i y_i) = \Sigma\,(xy)$,

the second moment is $\sum_{i=1}^{n} (x_i{}^2 y_i) = \Sigma\,(x^2 y)$, and so on.

(The moments are also defined as the foregoing sums divided by Σy. The use of this division would not change the equations which follow.)

If the observed values of y were precisely the same as the values computed from an empirical equation, the moments of both sets of values would be identical. If we impose the condition that these moments are equal, where the computed values are expressed in terms of the arbitrary constants to be determined, we secure a set of equations that will enable us to find the values of these constants. Clearly, we need as many moment equations as there are constants to be found.

For example, let us apply the method of moments to the data in the example of §XI.5. We shall need two moment equations, involving the zeroth and first moments. Using the symbolic form, these are,

$$\Sigma y = m\Sigma x + b\Sigma 1,$$
$$\Sigma xy = m\Sigma x^2 + b\Sigma x.$$

Note that, by our definition, $\Sigma 1 = n$, since each of the n terms is 1.

The details of the computation are shown in the following table:

x	y	x^2	xy
0.5	10	0.25	5.0
1.0	19	1.00	19.0
1.5	29	2.25	43.5
2.0	46	4.00	92.0
2.5	51	6.25	127.5
3.0	66	9.00	198.0
3.5	78	12.25	273.0
4.0	89	16.00	356.0
4.5	101	20.25	454.5
5.0	113	25.00	565.0
27.5	602	96.25	2133.5

$\Sigma 1 = n = 10$, $\Sigma x = 27.5$, $\Sigma x^2 = 96.25$, $\Sigma y = 602$, $\Sigma xy = 2133.5$. The two equations are

$$602 = 27.50m + 10.0b,$$
$$2133.5 = 96.25m + 27.5b.$$

Solving, we find $m = 23.18$ and $b = -3.53$, and the resulting equation is

$$y = 23.18x - 3.53.$$

While this method is slightly more involved than the method of averages, or the graphic method, it has the decided advantage of being independent of the individual judgment of the computer, so that the results are unique.

The extension of the method of moments to fitting polynomials of second and higher degrees is quite direct. For example, for the equation

$$y = ax^2 + bx + c,$$

the moment equations are

$$\begin{aligned} \Sigma y &= a\Sigma x^2 + b\Sigma x + c\Sigma 1, \\ \Sigma xy &= a\Sigma x^3 + b\Sigma x^2 + c\Sigma x, \\ \Sigma x^2 y &= a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2. \end{aligned}$$

Although the method of moments can be extended to any reasonable number of constants, the fact that moments of higher order tend to put undue stress on the values corresponding to the larger values of x makes the use of moments of order higher than 4 of questionable value.

EXERCISES

1. Determine by the method of moments the equations of straight lines fitting the following data:

(a)

x:	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y:	4	9	16	24	28	34	38	46.

(b)

x:	0	1	2	3	4	5	6	7	8
y:	12	24	30	43	50	58	64	75	87.

(c)

x:	20	25	30	36	40	45	50
y:	76	78	80	81	82	84	85.

(d)

x:	1	2	3	4	5	6	7	8	9	10
y:	16.4	15.0	13.8	12.3	10.5	8.8	7.4	6.1	4.9	3.2.

2. If a cubic centimeter of water at its greatest density, at 4° C., is heated, the increase in volume, h, expressed in units of 10^{-5} cc. is observed as follows:

t(° C.):	10	20	30	40	50	60	70	80	90	100
h(10^{-5} cc.):	12	177	435	782	1207	1705	2270	2899	3590	4343.

Determine an empirical equation of the form

$$h = a + bt + ct^2,$$

fitting these observations.

3. Compute the values of h in the preceding exercise, using the derived empirical formula. Compare the first and second moments of the computed and the observed values.

4. Derive an empirical formula for y in terms of x from the following data:

x:	1	2	3	4	5	6
y:	20.0	24.4	28.4	32.0	35.2	38.1.

5. Observed corresponding values of x and y are as follows:

x:	0	1	2	3	4	5
y:	4.7	14.7	21.7	25.8	24.8	19.5.

Derive a quadratic formula for these data, and compute values of y for

x: 0.5 1.5 2.5 3.5 4.5 6.0.

6. Compute the constants in Exercise 4 by the method of averages.
7. Compute the formula of Exercise 4 of §XI.5 by the method of moments.

XI.7 TRANSFORMATION OF DATA

The method of averages and the method of moments are especially convenient for the determination of empirical constants when the equations are linear in these constants. As we have seen in some particular cases, it may be possible, by means of a simple transformation, to reduce a nonlinear case to a linear case. We consider a few of these in detail.

(a) The exponential function $y = ab^x$

If we take the logarithms of both members of this equation, we obtain the new equation,

$$\log y = \log a + x \log b.$$

The substitution, $u = \log y$, $A = \log a$, $B = \log b$ changes this to

$$u = A + Bx,$$

which is the typical form for the use of the method of moments (or the method of averages).

EXAMPLE. The number of bacteria in a culture was observed at regular intervals of time to be as follows:

t:	0	1	2	3	4	5	6	7
N:	50	95	165	304	546	1012	1830	3330.

Compute an empirical formula for N as a function of t.

First construct a scatter diagram, using t and N as co-ordinates, and draw a smooth curve to indicate the trend. The character of this curve indicates the exponential form for the empirical formula. An examination of the data shows that successive values of N have ratios of 1.9, 1.74, 1.84, 1.79, 1.85, 1.81, and 1.82, so that the observed numbers approximate a geometric progression with ratio 1.8.

We assume, then, that

$$N = ab^t.$$

(There are also biological reasons for assuming this type of formula.)

The assumed equation may be changed to

$$\log N = \log a + t \log b.$$

Using the method of moments, we form equations from the zeroth and first moments:

$$\Sigma \log N = \log a \cdot \Sigma 1 + \log b \cdot \Sigma t,$$
$$\Sigma(t \log N) = \log a \cdot \Sigma t + \log b \cdot \Sigma t^2.$$

The details of the computation are given in the following table:

t	N	$\log N$	t^2	$t \log N$
0	50	1.6990	0	0.0000
1	95	1.9777	1	1.9777
2	165	2.2175	4	4.4350
3	304	2.4829	9	7.4487
4	546	2.7372	16	10.9488
5	1012	3.0051	25	15.0255
6	1830	3.2625	36	19.5750
7	3330	3.5224	49	24.6568
28		20.9043	140	84.0675

$\Sigma 1 = n = 8$, $\Sigma t = 28$, $\Sigma t^2 = 140$, $\Sigma \log N = 20.9043$, $\Sigma(t \log N) = 84.0675$, and the equations for the determination of $\log a$ and $\log b$ are

$$20.9043 = 8 \log a + 28 \log b,$$
$$84.0675 = 28 \log a + 140 \log b.$$

The solutions are

$$\log a = 1.7045, \log b = 0.2596,$$
$$a = 50.64, \qquad b = 1.818,$$

and hence our empirical equation is

$$N = 50.64\ (1.818)^t.$$

(b) The power function $y = ax^k$

If we take the logarithms of both members of this equation, we have

$$\log y = \log a + k \log x.$$

A substitution, $u = \log y$, $z = \log x$, $A = \log a$, gives

$$u = A + kz,$$

which is of linear form in A and k.

EXAMPLE. The quantity of water flowing through a pipe is known to be expressed by the equation

$$q = ar^k,$$

where q is the number of thousands of cubic feet per minute, r the radius of the pipe, a and k constants. Determine a and k from the following data:

r:	1	2	3	4	5	6
q:	4.8	27.5	75.2	152.4	264.2	410.0.

We first plot a scatter diagram from these data, using log r and log q as co-ordinates. Since the points tend to lie on a straight line, the prediction of a power function is verified.

Using the method of moments, we have

$$\Sigma \log q = \log a \; \Sigma 1 + k \Sigma \log r,$$
$$\Sigma[\log r \cdot \log q] = \log a \; \Sigma \log r + k \; \Sigma(\log r)^2.$$

The following is a convenient form for the computation

x	q	$\log r$	$\log q$	$(\log r)^2$	$\log r \cdot \log q$
1	4.8	.00000	.68124	.00000	.00000
2	27.5	.30103	1.43933	.09062	.43328
3	75.2	.47712	1.87622	.22764	.89518
4	152.4	.60206	2.18298	.36248	1.31428
5	264.2	.69897	2.42193	.48856	1.69286
6	410.0	.77815	2.61278	.60552	2.03313
		2.85733	11.21448	1.77482	6.36873

$n = 6$, $\Sigma \log r = 2.85733$, $\Sigma(\log r)^2 = 1.77482$, $\Sigma \log q = 11.21448$, $\Sigma(\log r \cdot \log q) = 6.36873$.

The equations are

$$11.21448 = 6 \log a + 2.85733k,$$
$$6.36873 = 2.85733 \log a + 1.77482k.$$

The solutions are

$$k = 2.48297, \qquad \log a = .68667, \qquad a = 4.86037.$$

The required equation is

$$q = 4.86037 \, r^{2.48297}.$$

(c) Equations of the hyperbolic type

An equation of the form $y = \dfrac{ax + b}{x + c}$, characterized by the fact that for a particular value of x, y becomes infinite, and that y approaches a limiting value as x increases indefinitely, may be fitted to appropriate data by the method of moments.

If we clear of fractions, the resulting equation,

$$xy + cy = ax + b,$$

is linear in the constants. Using the method of moments, we find three equations

$$\begin{aligned} \Sigma xy + c\Sigma y &= a\Sigma x + b\Sigma 1, \\ \Sigma x^2y + c\Sigma xy &= a\Sigma x^2 + b\Sigma x, \\ \Sigma x^3y + c\Sigma x^2y &= a\Sigma x^3 + b\Sigma x^2. \end{aligned}$$

From these equations it is possible to determine the values of a, b, and c.

We illustrate with an example that, although not precisely of the form indicated, will show the use of this method.

EXAMPLE. Gordon's formula for the critical pressure, p, for oak columns is

$$p = \frac{a}{b + R^2},$$

where R is the ratio of the length to the least dimension of a cross section, and a and b are constants. Determine values for a and b from the experimental data in the accompanying table:

R:	10	15	20	25	30	35	40
p:	845	770	675	585	510	435	375.

When cleared of fractions, the equation becomes

$$bp + pR^2 = a.$$

Applying the extended principle of moments,

$$\begin{aligned} b\Sigma p + \Sigma pR^2 &= na, \\ b\Sigma pR + \Sigma pR^3 &= a\Sigma R. \end{aligned}$$

The computation is as follows:

R	p	R^2	pR	pR^2	pR^3
10	845	100	8450	84500	845000
15	770	225	11550	173250	2598750
20	675	400	13500	270000	5400000
25	585	625	14625	365625	9140625
30	510	900	15300	459000	13770000
35	435	1225	15225	532875	18650625
40	375	1600	15000	600000	24000000
175	4195		93650	2485250	74405000

$$\Sigma R = 175,\ \Sigma p = 4195,\ n = 7,\ \Sigma pR = 93650,$$
$$\Sigma pR^2 = 2{,}485{,}250,\ \Sigma pR^3 = 74{,}405{,}000.$$

The equations are

$$4195\,b + 2{,}485{,}250 = 7\,a,$$
$$93{,}650\,b + 74{,}405{,}000 = 175\,a.$$

The solution is

$$b = 1{,}093.44,\quad a = 1{,}009{,}962.$$

Hence, we obtain on substitution,

$$p = \frac{1{,}009{,}962}{1{,}093.44 + R^2}.$$

EXERCISES

1. Given

x:	0	1	2	3	4	5
y:	2.11	2.44	2.75	3.08	3.30	3.87.

Assume that $y = ae^{bx}$, and compute a and b from these data.

2. The difference in temperatures of a heated body and of the air was observed at the end of each minute for 5 minutes, and the following data obtained:

t (time):	1	2	3	4	5	6
u (difference in temperature):	119	106	94	84	75	67.

By means of a scatter diagram, show that an empirical equation of the form $u = ae^{bt}$ may be used. Compute a and b.

3. Fit an empirical equation of the form

$$y = ax^b$$

to the following data:

x:	1	2	3	4	6	8
y:	7.2	10.4	12.8	15.0	18.3	21.1.

4. The following observations were made on the rate at which a wheel rotated in water after the power had been cut off:

t (number of seconds):	0	5	10	15	20	25
R (revolutions per minute):	1000	496	247	122	60	29.

Fit an empirical formula to the data.

5. The intensity of light that passes through a substance of thickness h centimeters is observed to be:

h:	0	5	10	15
L:	75.7	35.2	16.7	7.8.

Using these observations, determine the empirical constants a and b in the formula $L = ab^h$.

6. The difference between the observed value of a function and its computed value is called a **residual**. Calculate the residuals in Exercise 5.

7. Fit an empirical equation of the form $y = a + b/x$ to the following data:

x:	1	2	3	4	5
y:	20.2	13.3	11.1	10.0	9.3.

8. Determine a and b in the formula $y = \dfrac{ax}{b + x}$ from the following data:

x:	3	4	5	6	7	8
y:	15.5	17.4	19.3	20.6	21.8	22.6.

In the following exercises find an empirical equation fitting the given data:

9.

n:	1	2	4	6	8	10	12
P:	1.5	3.2	5.0	9.5	13.0	16.0	19.0.

P = brake horsepower of engines of a certain cylinder diameter, tabulated with the number of cylinders, n, of each engine.

10.

S:	10	20	30	40	50
R:	8	10	15	21	30.

R = resistance in pounds per ton to the motion of a train with speed S in miles per hour.

11.

p:	10	20	30	40	50	60	70
C:	307	275	249	224	204	185	169.

C (degrees centigrade) is the melting point for an alloy of zinc and lead, and p is the percentage of zinc in the alloy.

12.

t:	0	2	4	6	8	10	12
θ:	76.00	70.91	66.20	61.79	57.68	53.81	50.24.

θ = difference in temperatures of a cooling body and the surrounding air at the end of t minutes.

13.

p:	10	20	30	40	50	60
V:	37.78	19.73	13.50	10.39	8.35	6.63.

V = volume in cu. ft. of a pound of saturated steam at a pressure of p pounds per square inch.

CHAPTER

XII

RECTANGULAR CO-ORDINATES IN THREE DIMENSIONS

XII.1 INTRODUCTION

Any point in a three-dimensional space is determined by three co-ordinates.

If we were to locate an object in a room, let us say, a light, we might tell how far it is from each of two adjacent walls, and then tell its height from the floor. Again, to locate a collision between two airplanes, we would give the ground location, as in two-dimensional or plane geometry, and also the height at which the collision took place.

For such purposes we have, instead of straight lines of reference, three planes of reference. We do use three straight lines, called the co-ordinate axes, but they are simply the intersections of the reference planes. For convenience, we assume the reference planes to meet at right angles.

The distances from the three planes are called x,y,z, respectively. These are measured parallel to the corresponding axes, OX, OY, OZ.

A natural and pertinent question is, "How can we make a drawing on paper that will help us to visualize the situation as it exists?"

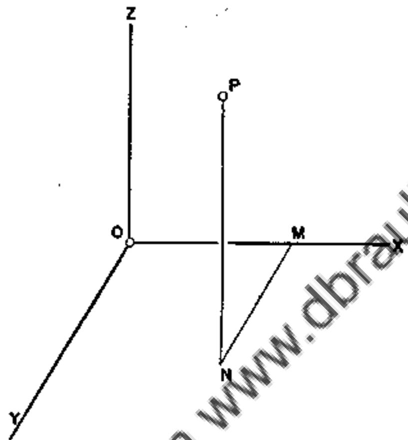

FIG. 92.—THREE CO-ORDINATES.

Consider Fig. 92, in which the x- and z-axes are assumed in the same relative positions as the x- and y-axes in plane geometry. A fair illustration is the corner of a room, in which we face the right wall, which represents the xz-plane. The floor represents the xy-plane. The intersection of the left wall and the floor is the y-axis. As we look at this, since it extends toward us, units of length on this axis appear shorter than equal units along either of the other axes.

A convenient drawing is one in which OY makes an angle of $120°$ with OX, and the units on the y-axis are half as long as the corresponding units on the x- and z-axes.

Then, to locate a point (x,y,z) we measure the distance $OM = x$, $MN = y$, parallel to the y-axis, OY (remember the foreshortening), and $NP = z$, parallel to the z-axis, OZ.

With a little practice this can be done very rapidly, and the imagination pictures the point P as standing out away from the planes.

It is essential to have all lines which are assumed parallel appear as parallel in the figure.

By common consent, unless it is explicitly stated otherwise, positive x is measured to the right, positive y toward the observer and positive z vertically upward.

EXAMPLE. Locate the three points, (3,4,2), (1,−1,−2), (−1,2,3), and draw the triangle having these points as vertices.

For the first point, we measure 3 units along OX, 4 units parallel to OY, and 2 units vertically upward.

For the second point, we measure 1 unit to the right, 1 unit parallel to OY, but in the negative direction, i.e., behind the xz-plane, or away from us, and 2 units vertically downward.

For the third point, we measure 1 unit to the left, 2 units in the positive y-direction, and 3 units vertically upward.

Joining these three points, we have the required triangle (Fig. 93).

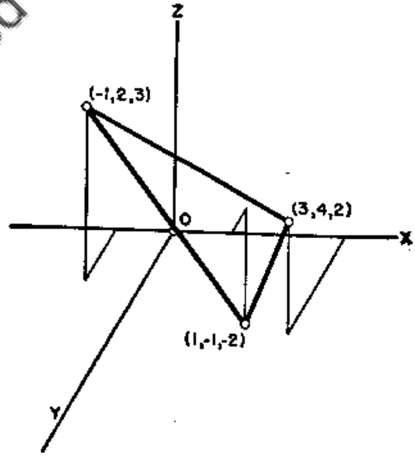

FIG. 93.

EXERCISES

Locate the following points by the method suggested in the text:

1. (1,2,3), (1,−2,−3), (−2,−2,3).
2. (3,4,−2), (−3,4,1), (3/2,−6,−2).
3. (0,3,1), (2,−4,0), ($\sqrt{13}$,0,0), ($-\sqrt{17}, -\sqrt{12}, -\sqrt{7}$).
4. Which of the co-ordinates of a point (x,y,z) must be zero if the point lies on the x-axis? On the y-axis? On the z-axis? In the xy-plane? In the yz-plane? In the xz-plane?
5. Make a table of signs of the co-ordinates of a point (x,y,z) for each of the eight octants.
6. From geometry we know that the foot, Q, of the perpendicular from a point, P, to a plane, is the projection of P on the plane. What are the co-ordinates of Q, if $P(x,y,z)$ be projected on (a) The xy-plane? (b) The yz-plane? (c) The xz-plane?
7. Read the co-ordinates of the projections of the points of Exercise 1 upon the xy-plane, of Exercise 2 on the yz-plane, and of Exercise 3 upon the xz-plane.

XII.2 DISTANCE BETWEEN TWO POINTS

Consider the two points, $P_1(2,2,2)$ and $P_2(5,4,4)$. If we pass planes through these points parallel to the reference planes, we have a parallelepiped, as shown in Fig. 94. Since RP_2 is per-

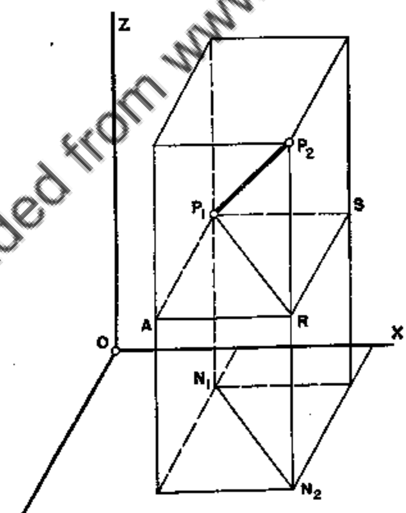

FIG. 94.—DISTANCE BETWEEN TWO POINTS.

pendicular to the plane P_1ARS, it is perpendicular to the line P_1R. Thus P_1RP_2 is a right triangle. Then, according to the Pythagorean theorem,

(1) $$\overline{P_1P_2}^2 = \overline{P_1R}^2 + \overline{RP_2}^2.$$

But

$$RP_2 = N_2P_2 - N_1P_1 = 2.$$

Hence

$$P_1R = N_1N_2 = (5-2)^2 + (4-2)^2 = 3^2 + 2^2 = 13.$$

Then

(2) $$P_1P_2^2 = 2^2 + 3^2 + 2^2 = 17.$$

This is an illustration of a general method of finding the distance between two points, of which, incidentally, the distance formula in two-dimensional geometry is a special case.

We think of the two points as $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$. The figure resembles that of the illustration. As before, the triangle P_1RP_2 is a right triangle.

(3) $$\begin{aligned}\overline{P_1P_2}^2 = \overline{P_1R}^2 + \overline{RP_2}^2 &= \overline{P_1S}^2 + \overline{SR}^2 + \overline{RP_2}^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.\end{aligned}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is the general formula for the distance between two points. If $z_1 = z_2$, the formula becomes exactly that for distance in two dimensions.

EXAMPLE. Show that the triangle A (3,4,2), B (1,−1,−2), C (5,1,3) is isosceles.

$$\overline{AB}^2 = (3-1)^2 + (4+1)^2 + (2+2)^2 = 45,$$
$$\overline{BC}^2 = (1-5)^2 + (-1-1)^2 + (-2-3)^2 = 45,$$
$$\overline{CA}^2 = (3-5)^2 + (4-1)^2 + (2-3)^2 = 14.$$

Since $AB = BC$, the triangle is isosceles.

EXERCISES

Locate the following pairs of points and find the corresponding distances:

1. (1,4,3), (5,2,7).
2. (−3/4,2,1), (5/4,−4,7).
3. (5,0,4), (−1,0,12).
4. (0,−9,−2), (0,0,−10).
5. Find the length d of the radius vector (distance from the origin) of the point (x_1,y_1,z_1).

6. (a) Locate the points $A(2,4,0)$, $B(5,4,0)$, $C(5,2,0)$, and find the co-ordinates of a point D, such that $ABCD$ is a rectangle.
(b) With this rectangle as a lower base, draw a parallelepiped $ABCDEFGH$ of height $AE = 6$, and find the lengths of BD and DF.
7. Show that the triangle $A(-2,1,3)$, $B(3,6,0)$, $C(4,-2,5)$ is scalene and acute.
8. Show that the triangle $A(-2,1,3)$, $B(0,1,1)$, $C(4,-2,5)$ is a right triangle.
9. By means of Formula (3) examine the following sets of points for collinearity. Draw the graphs.
(a) $(2,6,0)$, $(4,2,2)$, $(6,-2,4)$.
(b) $(0,4,5)$, $(6,0,5)$, $(9,-2,5)$.
(c) (a,b,c), $(2a,2b,2c)$, $(3a,3b,3c)$.
10. Locate the points $(1,6,5)$, $(4,2,7)$. Find the co-ordinates of the mid-point of the segment. Give reasons.
11. Derive formulas for the mid-point of segment (x_1,y_1,z_1), (x_2,y_2,z_2).
12. Find the lengths of the medians of the triangle $A(0,4,0)$, $B(8,0,0)$, $C(0,0,6)$.

XII.3 DIRECTION COSINES

The angle between two lines that do not meet is, by definition, the angle between two lines through a point and parallel to the given lines.

A directed segment P_1P_2 makes an angle $SP_1P_2 = \alpha$ with the x-axis, and angle $\beta = AP_1P_2$ with the y-axis, and an angle $\gamma = TP_1P_2$ with the z-axis (Fig. 95).

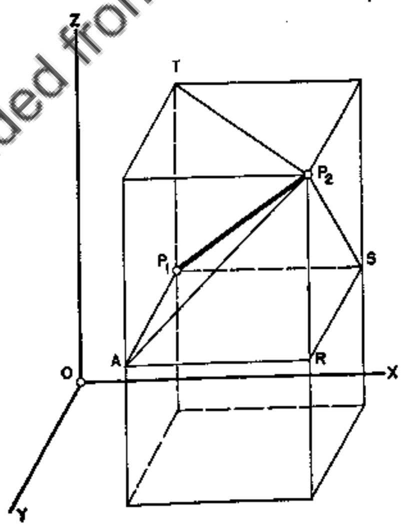

FIG. 95.—DIRECTION COSINES.

These three angles are called the **direction angles** of the line containing the segment P_1P_2. The angles P_1SP_2, P_1AP_2 and P_1TP_2 are all right angles. Then from trigonometry,

$$(1) \qquad \cos\alpha = \frac{P_1S}{P_1P_2} = \frac{x_2 - x_1}{P_1P_2},$$

$$\cos\beta = \frac{P_1A}{P_1P_2} = \frac{y_2 - y_1}{P_1P_2},$$

$$\cos\gamma = \frac{P_1T}{P_1P_2} = \frac{z_2 - z_1}{P_1P_2}.$$

We find these quantities, called the **direction cosines** of the line P_1P_2 more convenient for use than the actual angles, just as, in two dimensions, we found the slope more convenient than the direction angle. The direction cosines of P_2P_1 are negatives of those of P_1P_2.

If we square the members of (1) and add, we find the identity,

$$(2) \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma \equiv 1.$$

Evidently, then, the direction cosines of a line are not independent, but if any two are given, the third is determined except as to sign.

If the angle $\gamma = \pi/2$, the line is parallel to the xy-plane. But in that case, α and β are complementary angles, and hence $\cos\beta = \sin\alpha$. It is interesting to note that we might have used direction cosines in two-dimensional geometry instead of slopes. If this were done, we would see with added emphasis that two-dimensional geometry is a special case of three-dimensional geometry.

EXAMPLE. Find the direction cosines of the line joining the origin to (2,3,4).

$$\cos\alpha = \frac{2}{\sqrt{29}}, \cos\beta = \frac{3}{\sqrt{29}}, \cos\gamma = \frac{4}{\sqrt{29}}.$$

EXERCISES

Draw the lines determined by the following pairs of points, find the direction cosines, and indicate the direction angles in the drawing, the points being taken in the order given:

1. (0,0,0), (1,3,7).
2. (0,0,0), (−2,−4,5/2).

3. (2,6,5), (4,2,8).
4. (−3,4,7), (1,2,3).
5. What are the direction cosines of the x-axis? The y-axis? The z-axis?
6. What are the direction cosines of a line perpendicular to the xy-plane? The yz-plane? The xz-plane? The x-axis? The y-axis? The z-axis?
7. Can the direction cosines of a line be 1,1/3, and 1?
8. Find the remaining direction cosine:

   (a) $\cos\beta = 0$, $\cos\gamma = -\sqrt{3}/2$, α is acute.
   (b) $\cos\alpha = 1/3$, $\cos\beta = 1/4$, γ is obtuse.
9. On the line through (0,2,3) and (4,−2,5) determine a point for which x is 3.
10. What can you say of the direction cosines of a line parallel to the xy-plane? To the yz-plane? To the xz-plane? To the x-axis? To the y-axis? To the z-axis?
11. Using your knowledge of direction cosines, determine the position of the line through the following pairs of points, relative to the indicated plane or axis:

   (a) (3,5,2), (−5,6,2), xy-plane, z-axis.
   (b) (1,−4,7), (−1,−4,3), xz-plane, y-axis.
   (c) (3,5,2), (−5,5,2), x-axis, yz-plane.

XII.4 ANGLE BETWEEN TWO LINES

Consider two straight lines OP, OP', which intersect at the origin, making an angle θ (Fig. 96). Let the direction angles of the two lines be α, β, γ, and α', β', γ'. If PP' is a perpendicular drawn from P to the line OP', from the right triangle $OP'P$ we have

(1) $$OP' = OP\cos\theta.$$

The projection of P on the xy-plane is N, and the projection on

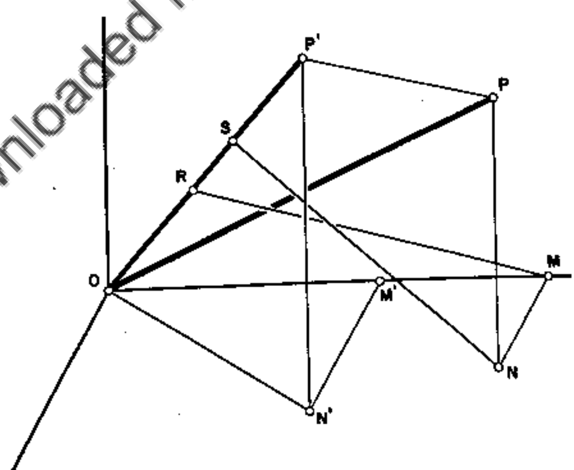

FIG. 96.—ANGLE BETWEEN TWO LINES.

the x-axis is M. Let the foot of the perpendicular from M to OP' be R, and let S be the foot of the perpendicular from N to OP'. According to trigonometry, the projection of any line segment on another line is the length of the segment times the cosine of the angle between the lines. Since OR, RS, and SP' are the projections of OM, MN, and NP, respectively, on OP', we have

(2)
$$OR = OM \cos \alpha',$$
$$RS = MN \cos \beta',$$
$$SP' = NP \cos \gamma'.$$

Note that MN is parallel to the y-axis, and NP is parallel to the z-axis. But

(3)
$$OM = OP \cos \alpha,$$
$$MN = OP \cos \beta,$$
$$NP = OP \cos \gamma.$$

Then, since O, R, S, P' are on the same line,

(4) $OP' = OR + RS + SP' = OM \cos \alpha' + MN \cos \beta' + NP \cos \gamma'$,

or, on substituting (1) and (3) in (4),

(5) $OP \cos \theta = OP \cos \alpha \cos \alpha' + OP \cos \beta \cos \beta' + OP \cos \gamma \cos \gamma'$.

Dividing through by the common factor, OP, we have,

(6) $$\cos \theta = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma'.$$

EXAMPLE. Show in two ways that the triangle $A(-4,2,-5)$, $B(4,2,-1)$, $C(2,-1,3)$ is a right triangle.

The direction cosines of AB are $8/\sqrt{80}$, 0, $4/\sqrt{80}$; those of BC are $2/\sqrt{29}$, $3/\sqrt{29}$, $-4/\sqrt{29}$; those of AC are $6/\sqrt{109}$, $-3/\sqrt{109}$, $8/\sqrt{109}$.

Formula (6) gives $\cos B = \dfrac{8\cdot 2 + 0\cdot 3 + 4(-4)}{\sqrt{80}\cdot\sqrt{29}} = 0$. Therefore B is a right angle.

For a second proof,

$$\overline{AB}^2 = 80,\ \overline{BC}^2 = 29,\ \overline{AC}^2 = 109;\ \overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2.$$

Therefore the triangle is a right triangle.

XII.5 PARALLEL AND PERPENDICULAR LINES

If two lines are parallel, they must have the same direction cosines. In order to avoid ambiguity, we may assume that the positive direction on any segment is that in which z is increasing. This would imply that $\cos \gamma$ is positive or zero. If z happens to be constant, we may choose the positive direction as that in which y increases.

By this convention, in the illustrative example of §XII.4, AC and AB are positive, but BC is negative.

This convention enables us to determine which of two supplementary angles we get when we apply Formula (6).

If two lines are perpendicular, $\cos \theta$ turns out to be zero. Conversely, if $\cos \theta = 0$, we know that the lines are perpendicular.

EXERCISES

Find the angle between the lines indicated by the following data:

1. $\cos \alpha = \frac{1}{2}$, $\cos \beta = \frac{\sqrt{3}}{2}$, $\cos \gamma = 0$,

   $\cos \alpha' = \frac{1}{\sqrt{3}}$, $\cos \beta' = \frac{1}{3}$, $\cos \gamma' = \frac{\sqrt{5}}{3}$.

2. $\cos \alpha = \frac{6}{7}$, $\cos \beta = \frac{-3}{7}$, γ is obtuse.

   $\cos \alpha = \frac{1}{\sqrt{5}}$, $\cos \beta = \frac{2}{\sqrt{5}}$.

3. (0,0,0), (4,2,4),
   (0,0,0), (−2,3,6).
4. (2,1,3), (4,−2,9),
   (−4,6,0), (0,0,12).
5. (−3,4,1), (1,−4,5),
   (0,0,0), (2,1,0).
6. Draw the quadrilateral $A(-3,4,5)$, $B(4,8,0)$, $C(7,4,2)$, $D(0,0,7)$, and show by means of direction cosines that the figure is a parallelogram.
7. Employing the theory of §XII.5, answer the following:
   (a) Is the figure of Exercise 6 a rectangle?
   (b) Is it a rhombus?
8. Using the direction cosines, show that the line segment DE between the mid-points of sides AB and BC of the triangle $A(0,6,6)$, $B(9,6,0)$, $C(5,-4,1)$ is parallel to AC and that the median through B is perpendicular to DE.
9. Show that $A(6,7,0)$, $B(3,1,-2)$, $C(8,4,6)$ are the vertices of a right triangle.
10. Show that $A(4,2,4)$, $B(10,2,-2)$, $C(2,0,-4)$ are the vertices of an equilateral triangle.

XII.6 VOLUME OF A TETRAHEDRON

Consider a tetrahedron $P_1P_2P_3P_4$, shown in Fig. 97. The vol-

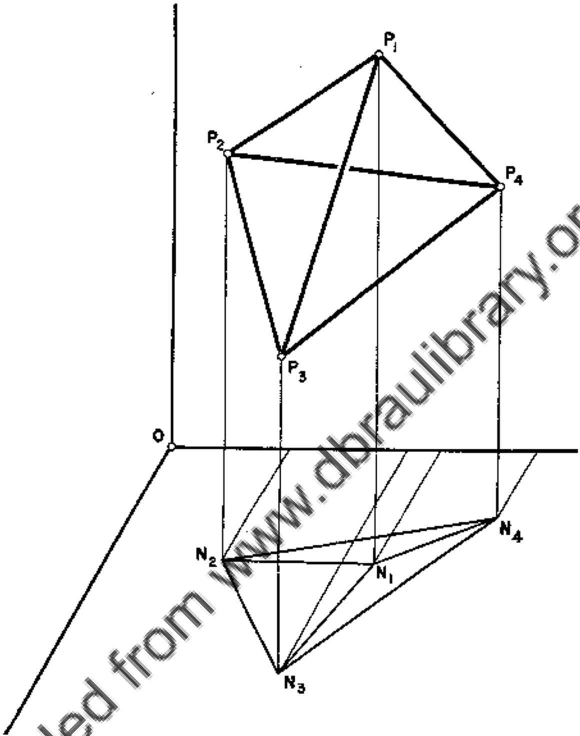

FIG. 97.—VOLUME OF A TETRAHEDRON.

ume of $P_1P_2P_3P_4$ can be found by adding the volumes of the truncated triangular prisms, $P_1P_2P_3N_1N_2N_3$, $P_1P_3P_4N_1N_3N_4$, $P_1P_4P_2N_1N_4N_2$ and subtracting the volume of $P_2P_3P_4N_2N_3N_4$.

The volume of a truncated triangular prism is the product of the area of a right section by one-third the sum of the parallel edges. Since the parallel edges are perpendicular to the xy-plane, the right sections are, $N_1N_2N_3$, $N_1N_3N_4$, $N_1N_4N_2$, $N_2N_3N_4$. The parallel edges are z_1, z_2, z_3, z_4. Then the volume of the first is,

(1) $$P_1P_2P_3N_1N_2N_3 = \frac{z_1 + z_2 + z_3}{3} \cdot \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Similarly for the other truncated prisms.

Then the desired volume is

$$(2)\quad P_1P_2P_3P_4 = \frac{z_1+z_2+z_3}{6}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \frac{z_1+z_3+z_4}{6}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} + \frac{z_1+z_4+z_2}{6}\begin{vmatrix} x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} - \frac{z_2+z_3+z_4}{6}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}.$$

But this is equivalent to the determinant

$$(3)\qquad \frac{1}{6}\begin{vmatrix} x_1 & y_1 & 1 & z_2+z_3+z_4 \\ x_2 & y_2 & 1 & z_1+z_3+z_4 \\ x_3 & y_3 & 1 & z_1+z_2+z_4 \\ x_4 & y_4 & 1 & z_1+z_2+z_3 \end{vmatrix}.$$

Interchanging the last two columns and changing signs in the new third column,

$$(4)\qquad \frac{1}{6}\begin{vmatrix} x_1 & y_1 & -z_2-z_3-z_4 & 1 \\ x_2 & y_2 & -z_1-z_3-z_4 & 1 \\ x_3 & y_3 & -z_1-z_2-z_4 & 1 \\ x_4 & y_4 & -z_1-z_2-z_3 & 1 \end{vmatrix}.$$

If we multiply the last column by $z_1 + z_2 + z_3 + z_4$ and add it to the third, the value of the determinant is the same and expresses the volume of the tetrahedron as

$$(5)\qquad P_1P_2P_3P_4 = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The student should carry through the proof for other special cases, such as when N_1 falls outside the triangle $N_2N_3N_4$.

If the four points lie in the same plane, the volume of the tetrahedron is zero. Hence the vanishing of the determinant in (5) may be taken as a condition that four points be coplanar. We sometimes say that then the points are **linearly dependent,** which simply means that all four sets of co-ordinates will satisfy the same linear equation in x, y, and z.

EXERCISES

Find the volume of the following tetrahedrons, whose vertices are the given points:

1. $(-1,2,3)$, $(5,1,0)$, $(2,4,8)$, $(3,-6,0)$.
2. $(0,0,0)$, $(5,8,0)$, $(5,0,6)$, $(0,4,3)$.
3. $(0,1,2)$, $(-4,5,0)$, $(3,3,11)$, $(2,5,12)$. Interpret the result.
4. Test for linear dependence. $(1,1,0)$, $(0,2,3)$, $(-2,-1,6)$, $(0,0,1)$.
5. Are the points $(0,0,0)$, $(-2,3,1)$, $(1,0,5)$ and $(-4,10,10)$ linearly dependent?
6. Find the value of x for which the points $(0,0,3)$, $(3,2,3)$, $(x,0,5)$, $(9,0,0)$ are coplanar.
7. Find values for x, y, and z, such that each of the corresponding points (x,y,z) are linearly dependent with $(0,1,2)$, $(5,1,0)$, $(4,3,1)$. Where are all these (x,y,z)-points located?

SUMMARY OF CHAPTER XII

Distance between two points P_1,P_2

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Direction cosines of a line P_1P_2

$$\cos \alpha = \frac{(x_2 - x_1)}{d}$$

$$\cos \beta = \frac{(y_2 - y_1)}{d}$$

$$\cos \gamma = \frac{(z_2 - z_1)}{d}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

Angle between two lines

$$\cos \theta = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2$$

Volume of a tetrahedron $P_1P_2P_3P_4$

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

CHAPTER

XIII

THE PLANE

XIII.1 PLANE THROUGH THREE POINTS

Three points not on a straight line determine a plane. If the volume of a tetrahedron is zero, we know that the four vertices lie in the same plane. Also, if the fourth point lies in the plane determined by the other three, the volume of the tetrahedron is zero.

Then we may say that a point (x,y,z) will lie in the plane determined by (x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3) if, and only if, the volume of the tetrahedron with these four vertices is zero. But, from §XII.6, this means that

$$(1) \qquad \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0,$$

and this is the equation of the plane.

EXAMPLE. Determine the equation of the plane through the three points (2,1,5), (3,2,1), (5,1,2).

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 5 & 1 \\ 3 & 2 & 1 & 1 \\ 5 & 1 & 2 & 1 \end{vmatrix} = x\begin{vmatrix} 1 & 5 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 5 & 1 \\ 3 & 1 & 1 \\ 5 & 2 & 1 \end{vmatrix} + z\begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 1 \\ 5 & 1 & 2 \end{vmatrix}$$

$$= -3x - 9y - 3z + 30 = 0.$$

If we divide through by -3, we have the equation

$$x + 3y + z - 10 = 0.$$

XIII.2 SKETCHING THE PLANE

In drawing the sketch of a plane, there are two principal cases, which are illustrated in the following examples. In one case the plane cuts the axes at points not at the origin and in the other the plane passes through the origin.

EXAMPLE 1. Draw the plane $3x + y + 2z = 6$.

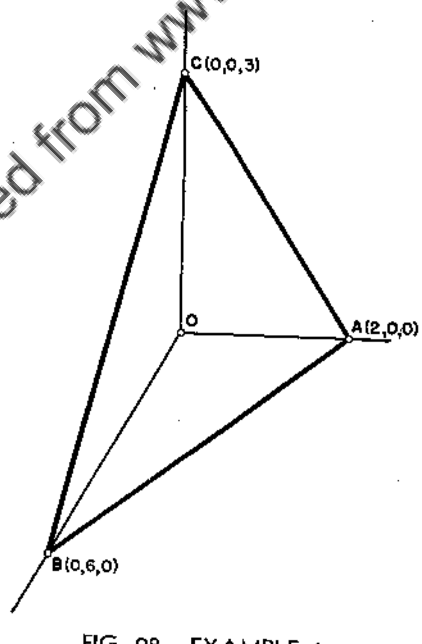

FIG. 98.—EXAMPLE 1.

Since the plane does not pass through the origin, the traces, or lines in which the plane cuts the planes of reference, show the plane very clearly (Fig. 98). These can be found by joining the points A, B, and C, in which the plane meets each of the co-ordinate axes.

EXAMPLE 2. Sketch the plane $x - 2y - z = 0$.

Since this plane passes through the origin, it is more difficult to see the plane as shown by its traces. The three-dimensional character can be brought out (as in Fig. 99) by drawing two of the traces and that in

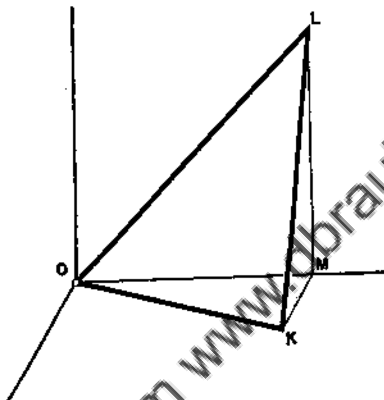

FIG. 99.—EXAMPLE 2.

which the plane cuts a plane parallel to one of the reference planes: OK, OL, and KL.

The equation of OK, the trace in the xy-plane, is obtained by setting $z = 0$ in the equation and the trace OL by setting $y = 0$.

EXAMPLE 3. Sketch the line of intersection of the planes

$$x + 2y + z - 4 = 0,$$
$$2x + y - z - 4 = 0.$$

The traces of the two planes in any co-ordinate plane intersect in a point of the common line. If we join these points we have the desired line, MPN (Fig. 100).

To obtain the co-ordinates of P, the point at which the line cuts the xy-plane, we set $z = 0$ in the two equations and solve the resulting equations simultaneously. M and N are found in a similar manner by setting $x = 0$ and $y = 0$, respectively.

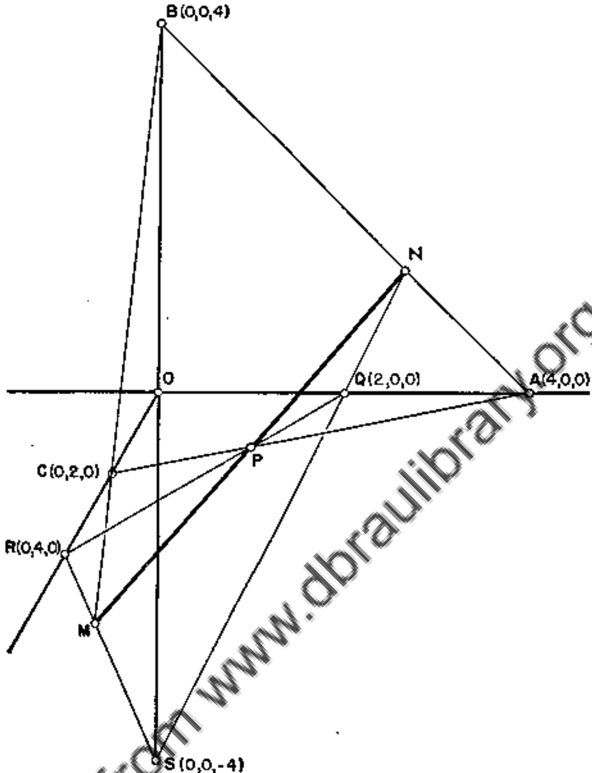

FIG. 100.—EXAMPLE 3.

EXERCISES

1. Write Equation (1) of §XIII.1 in the form $Ax + By + Cz + D = 0$. What determinants are represented by the coefficients?
2. Find the equations of the plane determined by the given points:
   (a) $\left(1,0,\frac{1}{2}\right)$, $(0,0,2)$, $(1,1,1)$.
   (b) $(2,1,-4)$, $(0,1,6)$, $(0,0,2)$.
   (c) $(1,1,0)$, $\left(-1,\frac{5}{2},0\right)$, $(5,-2,0)$.
3. Determine whether the following sets of points are coplanar:
   (a) $(0,1,-2)$, $(1,1,1)$, $(0,0,4)$, $(1,2,3)$.
   (b) $(2,1,4)$, $(0,0,1)$, $(1,1,3)$, $(2,2,5)$.

Sketch the following pairs of planes, and draw their intersections:

4. $x + 3y + 2z - 6 = 0$, $4x + 2y + z - 12 = 0$.
5. $2x - y + 4z - 8 = 0$, $x - y - 2z = 0$.
6. $4x + y - z = 0$, $x - 2y - 3z = 0$.
7. $2x + 3y + 6z = 6$, $3x + 2y + 6z = 6$.

7. Sketch the following planes, and describe their position relative to the co-ordinate planes:
   (a) $x = 0$, $y = 0$, $z = 0$.
   (b) $x - 4 = 0$, $y + z = 0$, $z - \frac{1}{2} = 0$.
   (c) $3x - y = 0$, $3x + 2y - 6 = 0$, $y - 2x + 4 = 0$.

XIII.3 NORMAL FORM OF THE EQUATION OF A PLANE

Consider the line, OP_1, of length p and having direction angles α, β, γ. The line, OP, where P is any point in the plane perpendicular to OP_1 at P_1, is projected into the line OP_1 (Fig. 101). (A line perpendicular to a plane is perpendicular to any line in the plane through the foot of the perpendicular.)

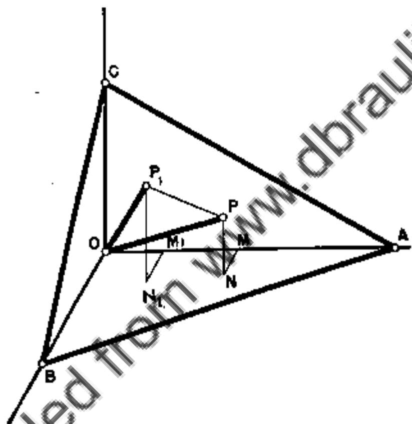

FIG. 101.—NORMAL EQUATION OF A PLANE.

If we apply the analysis of §XII.4, noting, as we do so, that $OM = x$, $MN = y$, $NP = z$, we have the equation

$$(1) \qquad x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

Since this condition must be satisfied by the co-ordinates of any point in the plane, it follows that (1) is the equation of the plane.

The observant student will notice the similarity between this development and that of the normal form of the equation of a straight line in two dimensions. We call this the **normal form** for the equation of the plane.

When $\gamma = \pi/2$, the z-term is missing, and then the equation **looks** exactly like the two-dimensional equation (replacing $\cos \beta$ by $\sin \alpha$, since $\alpha + \beta = \pi/2$). It must be interpreted, however, as the equation of a plane **parallel to the z-axis.**

To every plane there corresponds a unique set of direction cosines, namely, those of the normal, OP_1, drawn from the origin to the plane which may be called the direction cosines of the plane.

Since Equation (1) is of the first degree, and since the plane is any plane (p might be zero without affecting the analysis), we say that

The equation of any plane is of the first degree in x, y, and z.

EXAMPLE. Write the equation of the plane whose normal makes equal angles with the three axes and is of length $\sqrt{3}$. (Fig. 102.)

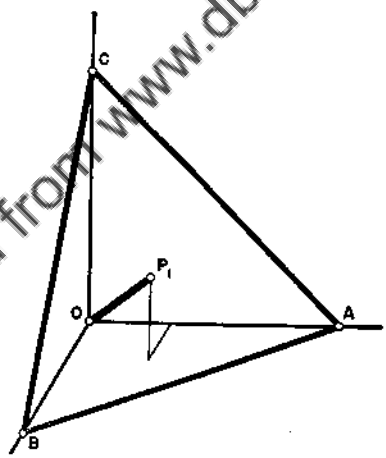

FIG. 102.

Since $\cos \alpha = \cos \beta = \cos \gamma$, and $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, we find that $\cos \alpha = 1/\sqrt{3}$.

Then the desired plane is

$$\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} - \sqrt{3} = 0,$$

or, if we clear of fractions,

$$x + y + z - 3 = 0.$$

EXERCISES

Write the normal form of the equation of the plane determined as indicated, and reduce to the general form $ax + by + cz + d = 0$:

1. $\alpha = 60°$, $\beta = 60°$, $\gamma = 45°$, $p = 5$.
2. $\alpha = 120°$, $\beta = -45°$, $\gamma = 60°$, $p = 4$.
3. $\alpha = 0°$, $\beta = 90°$, $\gamma = 90°$, $p = 3$.
4. $\alpha = 120°$, $\beta = -30°$, $\gamma = 90°$, $p = 5$.
5. $\alpha = 120°$, $\beta = 45°$, $\gamma = -60°$, $p = 3$.
6. $\alpha = \cos^{-1}\left(\frac{2}{3}\right)$, $\beta = \cos^{-1}\left(\frac{-1}{3}\right)$, $\gamma = \cos^{-1}\left(\frac{-2}{3}\right)$, $p = 5$.

From your knowledge of direction cosines first determine the position of the planes given by the following data relative to the co-ordinate planes and axes; then write the equation of the plane in the general form:

7. $\cos\alpha = \frac{3}{\sqrt{13}}$, $\cos\beta = \frac{2}{\sqrt{13}}$, $\cos\gamma = 0$, $p = \frac{6}{\sqrt{13}}$.
8. $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{2}$, $\gamma = 0$, $p = 3$.
9. $\alpha = 0$, $\beta = \frac{\pi}{2}$, $\gamma = \frac{\pi}{2}$, $p = 0$.
10. $\alpha = \frac{\pi}{2}$, $\beta = \cos^{-1}\left(\frac{-3}{5}\right)$, $\gamma = \cos^{-1}\left(\frac{4}{5}\right)$, $p = \frac{6}{5}$.
11. $\cos\alpha : \cos\beta : \cos\gamma = 1 : -4 : 8$, $p = 1$.

XIII.4 REDUCTION TO NORMAL FORM

Consider the most general equation of the first degree in x, y, and z:

$$Ax + By + Cz + D = 0. \tag{1}$$

If we can reduce this equation to the form (1) of §XIII.3, we know that it represents a plane.

Let us divide throughout by a constant k, to be determined:

$$\frac{A}{k}x + \frac{B}{k}y + \frac{C}{k}z + \frac{D}{k} = 0. \tag{2}$$

If this is in the normal form, the coefficients of x, y, and z are $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, respectively, and hence must satisfy the identity

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{3}$$

If we substitute and solve for k, we get

(4) $$k = \pm\sqrt{A^2 + B^2 + C^2}.$$

In conformity with our practice in two dimensions, we choose the sign of k opposite to that of D, so that OP_1 is always considered as a positive number. If D happens to be zero, we choose the same sign as that of C.

When we insert the value of k, Equation (1) is reduced to the desired form, and hence we know it represents a plane. We say, therefore,

Any equation of the first degree in x, y and z represents a plane.

The numbers A, B, C are proportional to the direction cosines of the normal line and are called **direction numbers of the normal to the plane.**

EXAMPLE. Reduce the equation

$$12x + 4y - 3z = 12$$

to the normal form.

Here $k = \sqrt{12^2 + 4^2 + 3^2} = \sqrt{169} = 13$. Hence, the desired equation is

$$\frac{12}{13}x + \frac{4}{13}y - \frac{3}{13}z - \frac{12}{13} = 0.$$

Hence we have

$$\cos\alpha = \frac{12}{13}, \cos\beta = \frac{4}{13}, \cos\gamma = -\frac{3}{13}, \text{ and } p = \frac{12}{13}.$$

XIII.5 DISTANCE OF A POINT FROM A PLANE

Two planes that are parallel have the same direction cosines for the normals except as to signs. The cosines of the two normals have the same signs when the planes lie on the same side of the origin and opposite signs when they lie on opposite sides of the origin.

Consider the plane

(1) $$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0$$

and the point (x_1,y_1,z_1), shown in Fig. 103.

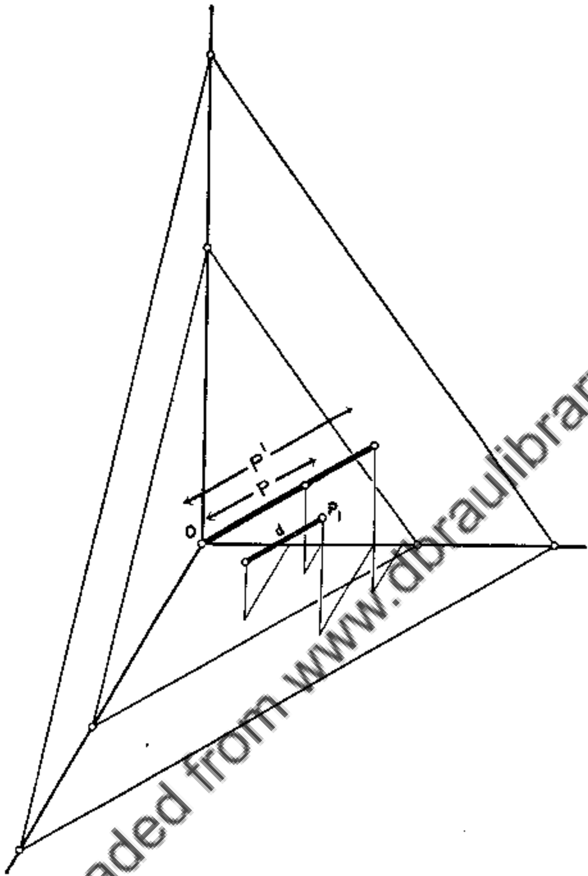

FIG. 103.

A plane parallel to (1) must have an equation of the form

(2) $$x \cos \alpha + y \cos \beta + z \cos \gamma - p' = 0,$$

where p' is the perpendicular distance from the origin to the new plane. Now let us make this new plane pass through the point (x_1,y_1,z_1). Then these co-ordinates must satisfy Equation (2), and we have

(3) $$p' = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma,$$

where p' is positive or negative accordingly as the planes are on the same or opposite sides of the origin.

The distance from plane (1) to plane (2), [and consequently to the point (x_1,y_1,z_1)] is

(4) $$d = p' - p = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p.$$

Then, as in the corresponding problem in two dimensions, to find the distance from the plane to a point, we need merely reduce the equation to the normal form and substitute the co-ordinates of the point.

EXAMPLE. Find the distance of the point $(3,2,-1)$ from the plane $3x - 4y + 6z + 12 = 0$.

Changing to normal form, $k = -\sqrt{61}$.

$$\frac{-3}{\sqrt{61}}x + \frac{4}{\sqrt{61}}y + \frac{-6}{\sqrt{61}}z - \frac{12}{\sqrt{61}} = 0.$$

Substituting the co-ordinates of the point, we find

$$d = \frac{-9}{\sqrt{61}} + \frac{8}{\sqrt{61}} + \frac{6}{\sqrt{61}} - \frac{12}{\sqrt{61}} = \frac{-7}{\sqrt{61}},$$

and, since the algebraic sign turned out to be negative, we know that the origin and the point $(3,2,-1)$ are on the same side of the plane.

EXERCISES

Reduce the following equations to the normal form and find the direction cosines and the length of the normal for each plane:

1. $x - 2y + 2z - 3 = 0.$
2. $2x + 3y + 6z - 4 = 0.$
3. $4y - 2z - 1 = 0.$
4. $x - y - z = 0.$
5. $2x - y + z = 1.$
6. $x + y + z = 1.$
7. $ax + by + cz + d = 0.$

Find the distance of the indicated point from the given plane and state whether this point and the origin are on the same or opposite sides of the plane. Draw the figure.

8. $(3,-2,2)$, $2x - y - z + 6 = 0.$
9. $(1,1,5)$, $2x + 3y - z - 4 = 0.$
10. $(10,0,3)$, $6x + 2y - 3z - 30 = 0.$
11. $(-6,0,2)$, $20x - 12y + 15z = 60.$
12. $(1,1,1)$, $2x + 3y - z - 4 = 0.$
13. $(0,0,0)$, $5x + 3y - 12z + 60 = 0.$
14. $(0,0,0)$, $x + y + z = 1.$
15. $(1,2,3)$, $3x + 2y + z = 0.$

XIII.6 ONE-POINT FORM OF EQUATION OF PLANE

Suppose a plane, with direction numbers A, B, C, passes through a point (x_1,y_1,z_1). The co-ordinates of the point must satisfy the equation of the plane, and so we may write

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

Hence

(1) $$Ax + By + Cz = Ax_1 + By_1 + Cz_1.$$

Sometimes the equation is written

(2) $$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,$$

although, in most cases the first form is more useful.

If instead of the direction numbers of the normal to the plane, we use the direction cosines, the equation is

(3) $$(x - x_1)\cos\alpha + (y - y_1)\cos\beta + (z - z_1)\cos\gamma = 0.$$

EXAMPLE. Write the equation of the plane with direction numbers of the normal 3, 4, 5, and passing through the point $(2, -1, 3)$.

The equation is, according to (1),

$$3x + 4y + 5z = 3\cdot 2 + 4(-1) + 5\cdot 3 = 17.$$

If we use (1) the form of the equation is

$$3(x - 2) + 4(y + 1) + 5(z - 3) = 0.$$

XIII.7 INTERCEPT FORM OF EQUATION OF PLANE

One of the simplest equations to write is the equation of a plane with known intercepts, i.e., a plane through the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$. It is

(1) $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

If we substitute the co-ordinates of these three points, we see that they satisfy the equation. We can, of course, derive the equation by applying the three-point form (§XIII.1).

The intercept form of the equation of the straight line in two-dimensional geometry is a special case of (1).

EXAMPLE. Write the equation of the plane with intercepts 2, 4, 5, respectively.

$$\frac{x}{2} + \frac{y}{4} + \frac{z}{5} = 1.$$

If we reduce to the general form, we have

$$10x + 5y + 4z - 20 = 0.$$

EXERCISES

In Exercises 1–8, the direction numbers of the normal to the plane (or the direction cosines) are given, as well as a point on the plane. Write the equation of the plane.

1. 1,3,5; (6,1,2).
2. 1,−3,4; (1,1,0).
3. 4,0,−2; (2,0,0).
4. 2,3,4; (0,3,−1/2).
5. 6,−1,0; (1/3,8,−3/4).
6. 1,−3,7; (0,0,0).
7. $1/2, 1/2, 1/\sqrt{2}$; (1,2,3).
8. 1/3,−2/3,2/3; (0,−1,6).
9. A line passes through the origin and the point $P(1,2,5)$. Write the equation of the plane perpendicular to this line at P. Reduce to the general form.
10. Find the equation of a plane through $P(3,0,4)$ and perpendicular to the line through P and the origin. Draw.
11. Draw the plane through $P_1(-3,-4,5)$ perpendicular to OP_1. Determine any other point, P_2, in the plane, and show that P_1P_2 is perpendicular to OP_1.

Find the intercept form of the equation of each of the following planes. Reduce to the general form. Check results.

12. x-intercept = 2, y-intercept = −3, z-intercept = 6.
13. x-intercept = 1, y-intercept = 2/3, z-intercept = −4/5.
14. x-intercept = p/q, y-intercept = r/s, z-intercept = q/p.
15. x-intercept = 5, y-intercept = −2, plane passes through (1,3,4).
16. Plane passes through (3,0,0), (0,2,0), (0,0,4).

XIII.8 THE STRAIGHT LINE

Consider the two equations,

(1)
$$\begin{aligned} x + 3y + z &= 6, \\ 3x + 4y - 2z &= 12. \end{aligned}$$

Any point whose co-ordinates satisfy both equations must be common to the two planes (ABC, DEF in Fig. 104) and hence is a point on the line of intersection (MPN). For this reason, we often speak of the two equations as **the equations of the line.**

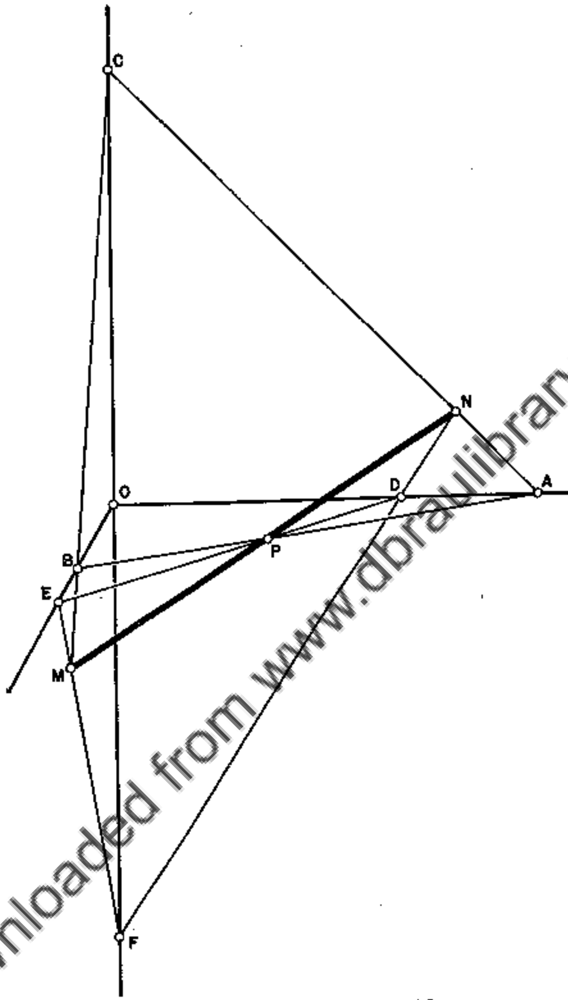

FIG. 104.—STRAIGHT LINE.

Thus, in Example 3, §XIII.1, we were drawing the line whose equations are

$$(2) \qquad \begin{aligned} x + 2y + z - 4 &= 0, \\ 2x + y - z - 4 &= 0. \end{aligned}$$

At times it is necessary to find the direction cosines of a line when the equations are given.

Since this line lies in the planes, it is perpendicular to the normals to these two planes. If r, s, t are the direction numbers of the line

(2)
$$\begin{aligned} x + 2y + z - 4 &= 0, \\ 2x + \ \ y - z - 4 &= 0, \end{aligned}$$

the condition for perpendicularity (§XII.5) gives us two equations:

(3)
$$\begin{aligned} 1\,r + 2\,s + 1\,t &= 0, \\ 2\,r + 1\,s - 1\,t &= 0. \end{aligned}$$

From these we find, by solving for $\frac{r}{t}, \frac{s}{t}$

(4) $$r : s : t = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} : -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} : \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 : 3 : -3 = 1 : -1 : 1.$$

In general, the direction numbers r, s, t of a line

(5)
$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0, \\ A_2x + B_2y + C_2z + D_2 &= 0, \end{aligned}$$

are given by

(6) $$r : s : t = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} : -\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} : \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

XIII.9 INTERSECTION OF THREE PLANES

In general, three planes determine a point. Finding the point when the equations of the planes are given, is a problem in simultaneous linear equations in algebra.

The cases that are labelled inconsistent in algebra are those in which the three planes are all parallel to a single line, thus forming the three faces of a triangular prism, and hence have no finite point in common, or two of the planes are parallel but not parallel to the third.

EXAMPLE. Examine the relationship of the three planes

$$\begin{aligned} x + \ \ y + 2z &= 6, \\ 3x + 2y - 2z &= 6, \\ 4x + 3y + 0z &= 6. \end{aligned}$$

If we solve for x, by determinants, we formally obtain

$$x = \frac{\begin{vmatrix} 6 & 1 & 2 \\ 6 & 2 & -2 \\ 6 & 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 3 & 2 & -2 \\ 4 & 3 & 0 \end{vmatrix}} = \frac{36}{0}.$$

Hence the equations are inconsistent, since no value exists for x.

Graphing, we find the three lines of intersection are parallel (Fig. 105). The plane ABC, represented by the first equation, intersects the other

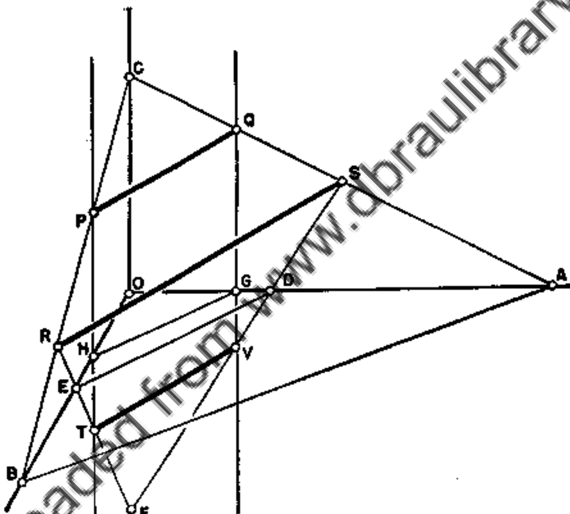

FIG. 105.

two planes, DEF and $PQHG$, in the parallel lines PQ and RS. Likewise, TV is the intersection of the planes DEF and $PQHG$.

EXERCISES

1. Represent each of the co-ordinate axes by the equations of two planes, taken simultaneously.

Find the direction numbers of each of the lines represented by the following pairs of planes:

2. $x + y + z - 2 = 0$, $2x - 3y + z + 3 = 0$.
3. $2x + y - 6z - 4 = 0$, $x - 3y + 5z - 1 = 0$.
4. $3x + y - 4z = 2$, $2y + 5z = 1$.
5. $x + 2y - 7z = 0$, $2x - y + 7z = 3$.
6. Find the direction cosines of the lines of Exercises 2–5.

7. Check the results of Exercises 4 and 5, by finding two points on each line and, from them, the direction cosines of the line.
8. Do the planes $x + 2y + 3z = 0$, $x + y + z - 1 = 0$, and $2x - y = 3z$, have a common point? Do they have more than one common point?
9. If we denote the equations of Example 1, §XIII.9, as (1), (2), and (3), respectively, show that the lines represented by (1) and (2), by (1) and (3), and by (2) and (3) are parallel, by finding the direction cosines.
10. Determine the relation between the following sets of planes:

    (a) $x + 2y = 0$, $y = \frac{3z}{2}$, $x = 3z$.

    (b) $2x - 3z = 6$, $x + 2z - 4 = 0$, $3x + 4y = 12$.
11. Show that the three planes, $2x - 3y + 5z - 2 = 0$, $x + \frac{y}{4/5} + \frac{z}{2/5} = 1$, and $x + \frac{y}{2/3} + \frac{z}{2/5} = 1$, intersect in a line.

XIII.10 EQUATION OF A STRAIGHT LINE—SYMMETRICAL FORM

Consider the straight line (Fig. 106) passing through the point (2,1,3) and having the direction numbers 2,3,2.

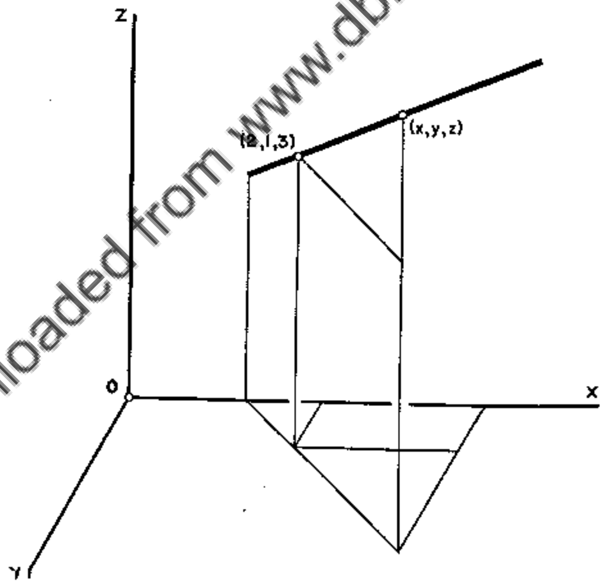

FIG. 106.—STRAIGHT LINE.

The direction numbers of a line (proportional to the direction cosines) are proportional to the differences of the co-ordinates of any two points of the line. Suppose the second point, any point of the line, to be (x,y,z). Then we must have,

$$(1) \qquad \frac{x-2}{2} = \frac{y-1}{3} = \frac{z-3}{2}.$$

This is known as the **symmetrical form** for the equations.

In general, if the point is (x_1,y_1,z_1), and the direction numbers are r, s, t, the symmetrical form for the equations of the line gives

$$(2) \qquad \frac{x-x_1}{r} = \frac{y-y_1}{s} = \frac{z-z_1}{t}.$$

In place of the direction numbers, we can, if necessary, use the actual values of the direction cosines:

$$\cos\alpha = \frac{r}{\sqrt{r^2+s^2+t^2}},$$

$$\cos\beta = \frac{s}{\sqrt{r^2+s^2+t^2}},$$

$$\cos\gamma = \frac{t}{\sqrt{r^2+s^2+t^2}}.$$

EXAMPLE. Reduce the equations of the line

$$2x - y + z + 4 = 0,$$
$$x + y + z + 1 = 0,$$

to the symmetrical form.

By the method of §XIII.7 we find the direction numbers:

$$r : s : t = -2 : -1 : 3.$$

To find a definite point, choose a value, say, $x = 1$ (any value will do), then

$$y - z = 6,$$
$$y + z = -2.$$

If we solve for y and z, $\quad y = 2, \quad z = -4.$

Then the equations are

$$\frac{x-1}{-2} = \frac{y-2}{-1} = \frac{z+4}{3}.$$

EXERCISES

1. In the example of this section the result is
$$\frac{x-1}{-2} = \frac{y-2}{-1} = \frac{z+4}{3}.$$
State reasons why the denominators $-2, -1, 3$ cannot be the direction cosines of the line. What are the direction cosines of the line?

2. Find the direction cosines of each of the following lines:

(a) $\dfrac{x+1}{3} = \dfrac{y}{1} = \dfrac{z+2}{-2}$.

(b) $\dfrac{x-4}{\frac{1}{2}} = \dfrac{y+2}{-\frac{1}{3}} = \dfrac{z}{\frac{2}{3}}$.

(c) $\dfrac{x-a}{p} = \dfrac{y+b}{q} = \dfrac{z-c}{\sqrt{1-p^2-q^2}}$.

Write the equations of the following lines in symmetric form.

3. Passing through $(1,2,3)$ and $(4,-1,0)$.
4. Passing through the origin and $(\frac{1}{2}, \frac{2}{3}, \frac{3}{4})$.
5. Passing through $(3,2,0)$ and having direction numbers $1, 5, -8$.
6. Passing through $(4,0,1)$, $\alpha = 45°$, $\beta = 60°$, $\gamma = 60°$.
7. $2x + y + z + 1 = 0$, $3x + 2y - z - 4 = 0$.
8. Find the angle between the following pairs of lines:

(a) $x + y + z - 2 = 0$, $2x - y + z + 1 = 0$, and $3x + y - z = 0$, $x - 3y + z = 0$.

(b) $\dfrac{x-1}{1} = \dfrac{y}{-2} = \dfrac{z+2}{5}$ and $x = y = -z$.

9. Find the equation of the plane through $(1,0,3)$ perpendicular to the line $x = y = 2z$.
10. Find the equation in symmetric form of the line through $(0,1,2)$ perpendicular to the plane $3x - 4y + 5z - 1 = 0$.

REVIEW EXERCISES CHAPTERS XII AND XIII

Draw the tetrahedron whose vertices are A (2,4,6), B (0,6,0), C (6,12,0), D (3,0,0), and use the drawing in working out the following exercises.

1. Find the distance from B to the centroid of triangle ACD.
2. Find angle B of triangle ABC.
3. Find the equation of the plane ABC in general, intercept, and normal form.
4. Find the volume of the tetrahedron.
5. Find the distance of B from the plane ACD. Find the area of triangle ACD (make use of result of Exercise 4).
6. (a) Write the equation of the line AB. (Two planes.)
   (b) Find the equation of line AB in symmetric form.
7. Write the equation of the plane through A perpendicular to the line AB.
8. Find the equation of the plane through D perpendicular to the line AB.
9. If you were told that the equations of the planes ABC, ACD, ABD, respectively, are $3x - 3y - 2z + 18 = 0$, $12x - 3y + 4z - 36 = 0$, and $6x + 3y - z - 18 = 0$, how would you check the truth of the statement? Show that these three planes intersect in a point.
10. Find the equation of the plane through
    (a) line AB and the midpoint of CD.
    (b) line AC and the midpoint of DB.
    (c) line AD and the midpoint of BC.
11. Show that, in Exercise 10, the line of intersection of planes (a) and (b) is precisely the line of intersection of planes (b) and (c), and hence that this line is common to all three planes.

SUMMARY OF CHAPTER XIII

Plane through 3 points

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Normal form—equation of plane

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p$$

Distance of point P_1 from plane

$$d = x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p$$
$$= \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}$$

One-point form—equation of plane

$$(x - x_1) \cos \alpha + (y - y_1) \cos \beta + (z - z_1) \cos \gamma = 0$$

or
$$A (x - x_1) + B (y - y_1) + C (z - z_1) = 0$$

Intercept form—equation of plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Straight line

$$A_1x + B_1y + C_1z + D_1 = 0$$
$$A_2x + B_2y + C_2z + D_2 = 0$$

Direction cosines

$$\cos \alpha : \cos \beta : \cos \gamma = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} : - \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} : \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

Symmetrical form

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}$$

CHAPTER

XIV

QUADRIC SURFACES

XIV.1 INTRODUCTION—THE SPHERE

In the last two chapters we have seen repeated examples showing that the geometry of two dimensions is a special case of three-dimensional geometry. Loci of the second degree, generally known as **quadrics,** offer no exception to this unity of form and idea.

For example, consider the locus of a point that lies at a fixed distance r from a fixed point (a,b,c). If we apply the distance formula from chapter §XII, we have the equation

$$(1) \qquad (x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

From our knowledge of geometry, we know that this locus is a sphere, of radius r with center at (a,b,c). Any equation of the form (1) represents a sphere, since it states that the distance of the point, (x,y,z), from the fixed point (a,b,c) is constant.

EXAMPLE. Analyze the locus whose equation is

$$3x^2+3y^2+3z^2-6x-12y-5z+3=0.$$

If we divide by 3 and complete squares, we have,

$$(x-1)^2+(y-2)^2+\left(z-\frac{5}{6}\right)^2=\frac{169}{36}.$$

Therefore, this equation represents a sphere with center at (1, 2, 5/6) and radius $\frac{13}{6}$. It intersects the xy-plane in a circle with center at $(1,2,0)$ and radius 2 (Fig. 107).

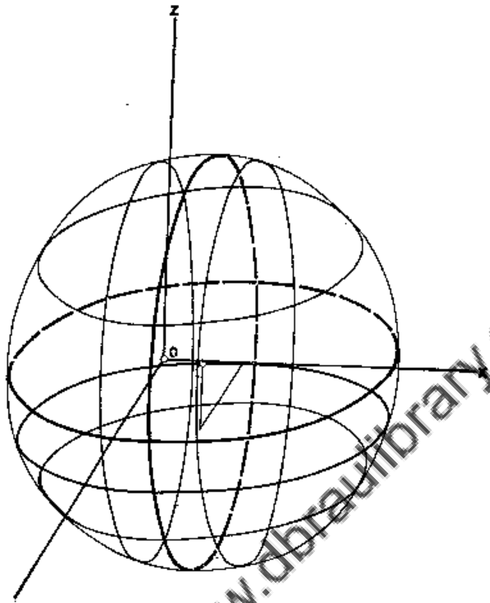

FIG. 107.—SPHERE.

EXERCISES

Write the equation of the sphere in each of the following exercises in the form (1) and reduce to the general form:

1. Center $(1,4,-3)$, radius $= 6$. 2. Center $(0,0,0)$, radius $= 4$.
3. Center $(0,0,0)$, passing through $(1,3,7)$.
4. Segment between $(-3,4,5)$ and $(7,0,3)$ is a diameter.
5. Radius $= 5$, sphere tangent to xy-plane at the origin.
6. Center at $(2,4,6)$, sphere tangent to yz-plane.
7. Center at $(1,2,5)$, sphere tangent to the plane $2x + 3y + 6z - 10 = 0$.
8. Center at (h, k, l), radius $= r$.

Find the center and radius of each of the following spheres; sketch, showing, in particular, the traces in the co-ordinate planes.

9. $x^2 + y^2 + z^2 = 25$. 10. $x^2 + y^2 + z^2 - 8y - 10z - 8 = 0$.
11. $x^2 + y^2 + z^2 - 8x - 4y - 0.25 = 0$.

In Exercises 12–14, determine, without first drawing, the position of the given point with respect to the given sphere. Then sketch.

12. $(0,-4,9)$, $(x - 6)^2 + (y - 4)^2 + (z - 5)^2 = 4$.
13. $(1,-2,-4)$, $x^2 + y^2 + z^2 - 4x + 6y = 12$.
14. $(0,0,4)$, $x^2 + y^2 + z^2 - 16 = 0$.
15. Show that the distance between the centers of
$$x^2 + y^2 + z^2 - 10x + 4y - 6z + 29 = 0$$
and $x^2 + y^2 + z^2 - 1 = 0$ is greater than the sum of the radii. What is the relative position of the spheres?

XIV.2 SKETCHING A QUADRIC SURFACE

In sketching a plane, we found that it was sufficient, ordinarily, to show the traces in three planes. For a more general surface, something more is required, just as we needed more than two points to show the shape of a conic section.

Naturally, one of the first steps is to find the traces in the co-ordinate planes. Then we draw the traces in planes parallel to the co-ordinate planes. We can best illustrate the process with an example.

EXAMPLE. Sketch the graph of the equation

$$4x^2 + 9y^2 + 16z^2 = 144.$$

Clearly x^2 cannot be greater than 36, since this would demand negative values for either y^2 or z^2. Similarly, y^2 cannot exceed 16, and z^2 cannot exceed 9. Every section of the surface by a plane is an ellipse. Hence the surface is called an **ellipsoid** (Fig. 108).

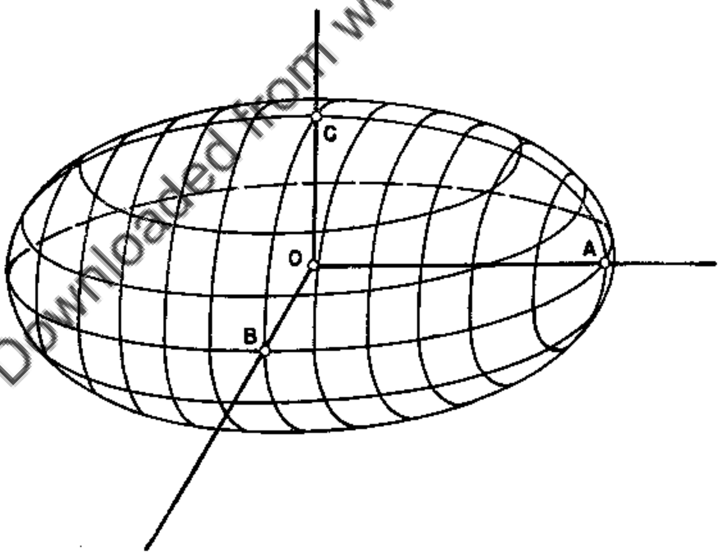

FIG. 108.—ELLIPSOID

The traces in the three co-ordinate planes are the ellipses

$$4x^2 + 9y^2 = 144,$$
$$4x^2 + 16z^2 = 144,$$
$$9y^2 + 16z^2 = 144.$$

Traces in parallel planes can be plotted from the equations found by substituting numerical values for x, y, z, respectively. If we recall that each such section is an ellipse with the ends of the major and minor diameters on the traces in the co-ordinate planes, we can locate these points graphically, and make the sketch with reasonable ease and speed.

EXERCISES

1. Given the equation $4x^2 + 9y^2 + z^2 = 36$,
   (a) Identify the quadric.
   (b) Find the x, y-, and z-intercepts and the lengths of the principal diameters.
   (c) Find the traces in the co-ordinate planes.
   (d) Find the traces in the planes $x = 1$, $x = -1$, $y = 1$, $y = -1$, $z = 1/2$, $z = -1/2$.
   (e) Sketch the surface.

By the method of the text, discuss the following surfaces, observing steps (a) to (e) of Exercise 1.

2. $4x^2 + 16y^2 + z^2 - 16 = 0$.
3. $9x^2 + 4y^2 + z^2 = 36$.
4. $x^2 + 9y^2 + 4z^2 - 36 = 0$.
5. $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{4} = 1$.
6. $\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{4} = 1$.
7. $100x^2 + 1600y^2 + 256z^2 - 6400 = 0$.

XIV.3 SHAPE OF QUADRICS—CENTRAL QUADRICS

If a central quadric is symmetrical with respect to the co-ordinate planes, x, y, and z occur only in even powers. Thus the central quadrics have a standard form,

(1) $$ax^2 + by^2 + cz^2 + d = 0,$$

where a, b, c, and d are all different from zero.

We can identify the various central quadrics by the relative values of a, b, c, and d. As was shown in the example of XIV.2, if a, b, c are all positive, and d is negative, the surface is an ellipsoid.

If a, b, c, d are all positive, there will be no real points on the surface, which we call an **imaginary ellipsoid**.

If one or two of the first three terms in (1) is negative, the surface is known as a **hyperboloid.**

Let us assume, for the sake of convenience, that a is positive and d negative. Then there will be two possible cases.

Case 1. The Hyperboloid of One Sheet

Suppose b is positive and c is negative. (The opposite choice of signs, with b negative and c positive, does not affect the essential analysis). We shall illustrate with a numerical example.

EXAMPLE. 1. Discuss and sketch the quadric surface

$$\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{16} = 1.$$

This surface does not intersect the z-axis. Any section parallel to the xy-plane is an ellipse, similar to the ellipse $x^2/9 + y^2/4 = 1$. Any section by a plane through the z-axis is a hyperbola. Since the surface consists of a single infinite part, it is called the **hyperboloid of one sheet** (Fig. 109).

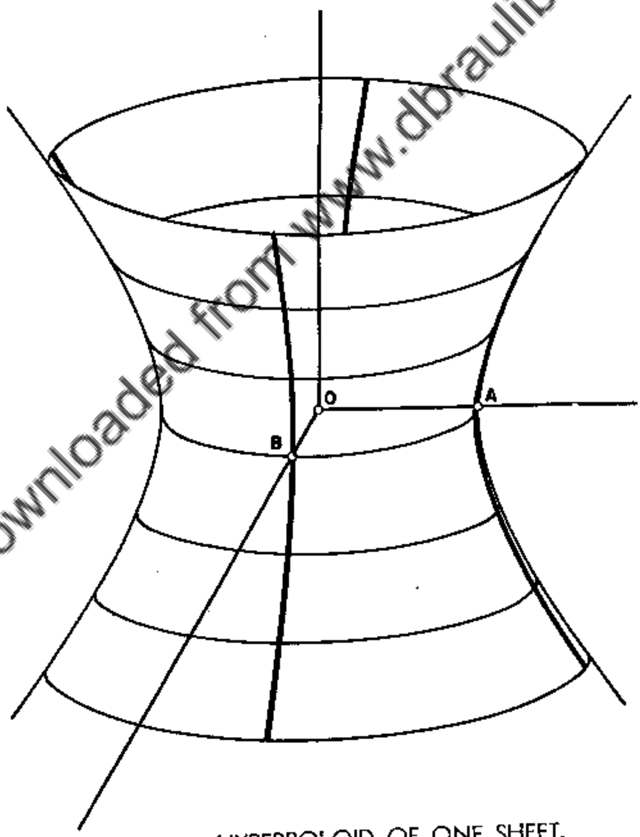

FIG. 109.—HYPERBOLOID OF ONE SHEET.

Case 2. The Hyperboloid of Two Sheets

Suppose that b and c are both negative. Here again, we illustrate with a numerical example, leaving the student to make the obvious generalization.

EXAMPLE 2. Discuss and sketch the quadric surface

$$4x^2 - 9y^2 - 9z^2 = 36.$$

There are no real points of the surface for which x^2 is less than 9. Every section by a plane parallel to the yz-plane is a circle (in the general case it would be an ellipse). The traces in the xz-plane and the yz-plane are hyperbolas (Fig. 110). As we suggested in the case of

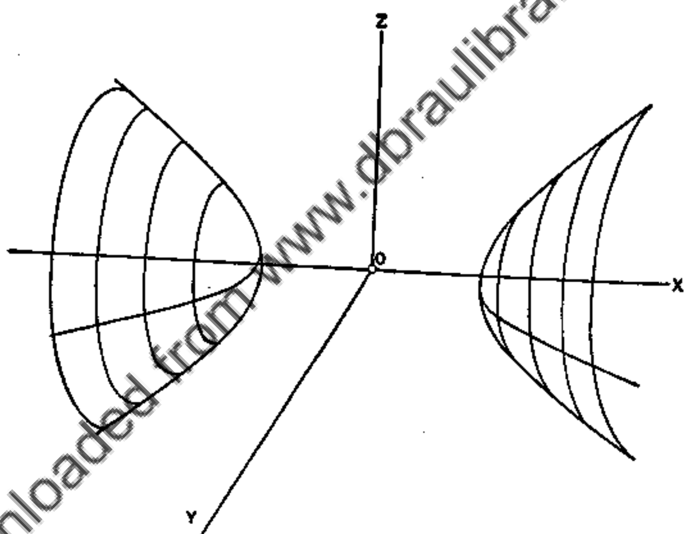

FIG. 110.—HYPERBOLOID OF TWO SHEETS.

the ellipsoid, sections parallel to the yz-plane can be plotted from the equations found by substituting values of x, or the graphic plan suggested there can be used.

As we noted before, any variation of signs in the equations of the hyperboloid of one sheet and the hyperboloid of two sheets, without changing the number of positive and negative signs, does not affect the essential character of the surface, but merely the orientation with respect to the co-ordinate system.

EXERCISES

Identify and sketch the following surfaces:

1. $\frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{4} = 1.$
2. $\frac{x^2}{4} - \frac{y^2}{25} + \frac{z^2}{9} = 1.$
3. $\frac{x^2}{4} - \frac{y^2}{25} - \frac{z^2}{9} = 1.$
4. $x^2 - y^2 + z^2 = 4.$
5. $x^2 - y^2 - z^2 = 4.$
6. $-x^2 + y^2 + z^2 = 25.$
7. $4x^2 + 3y^2 - 12z^2 = 48.$
8. $4x^2 + 3y^2 - 12z^2 = 48.$
9. $100x^2 - 4y^2 - 25z^2 + 100 = 0.$
10. $x^2 + y^2 - 4z^2 = 1.$

XIV.4 SHAPE OF QUADRIC—NONCENTRAL QUADRICS

A noncentral quadric is called a **paraboloid.** Of these there are two types. Any such surface has two planes of symmetry, which, for convenience we will assume to be the xz- and yz-planes. Then if the surface passes through the origin, the equation will appear in the form

$$(1) \qquad ax^2 + by^2 + nz = 0,$$

Where a, b, and c are all different from zero.

The two essentially different cases appear when we assume a and b to have the same or opposite signs. As in the case of the hyperboloids, we will illustrate each of these two cases with a numerical example.

Case 1. The Elliptic Paraboloid

Discuss and sketch the surface.

$$4x^2 + y^2 = 4z.$$

The surface is symmetrical with respect to the xz- and yz-planes, since x and y occur only in even powers. Any section parallel to the xy-plane is an ellipse, real if z is positive, imaginary if z is negative.

The surface lies wholly on one side of the xy-plane and extends indefinitely in the positive direction of the z-axis. The z-axis intersects the surface at the **vertex.**

Any section by a plane through or parallel to the z-axis is a parabola. There is no plane that cuts the surface in a hyperbola. This surface is known as the **elliptic paraboloid** (Fig. 111).

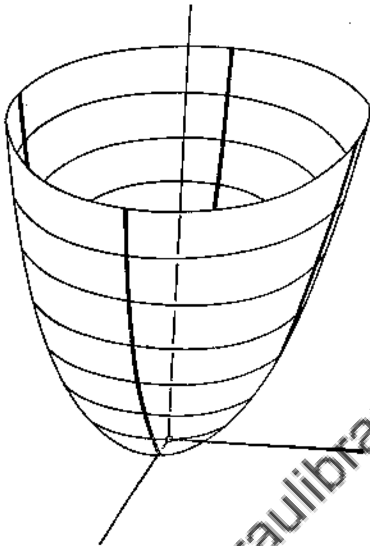

FIG. 111.—ELLIPTIC PARABOLOID.

If $a = b$, equation (1) can be reduced to the form

$$x^2 + y^2 = mx.$$

It is obvious that any section made by $z = k(k > 0)$ is a circle. The surface is called **paraboloid of revolution** since it could be generated by rotating a parabola about its axis.

Case 2. The Hyperbolic Paraboloid

Discuss and sketch the surface,

$$x^2 - y^2 = 4z.$$

Since x and y occur only in even powers, the surface is symmetrical with respect to the xz- and yz-planes.

Any section by a plane through or parallel to the z-axis is a parabola, since z occurs only in the first degree.

All sections parallel to the xz-plane are equal parabolas.

Any plane parallel to the xy-plane cuts the surface in a hyperbola whose asymptotes are parallel to the lines whose equations are given by

$$x^2 - y^2 = 0.$$

The xy-plane itself cuts the surface in the two lines,

$$x + y = 0, \quad z = 0. \qquad x - y = 0, \quad z = 0.$$

Since every section of the curve is either a hyperbola or a parabola, the surface is called the **hyperbolic paraboloid** (Fig. 112).

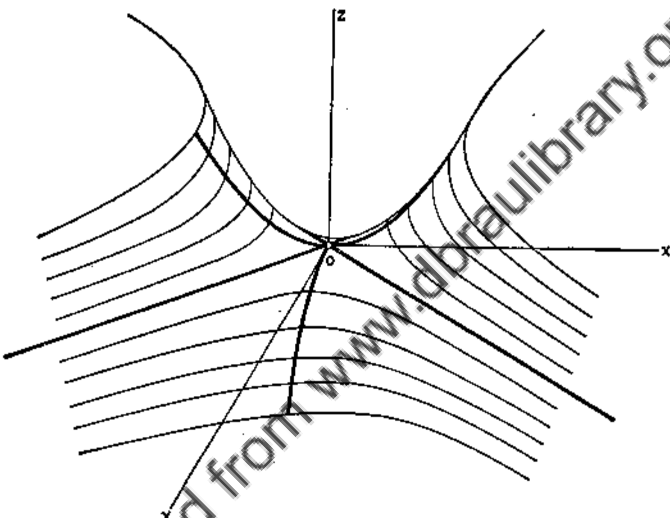

FIG. 112.—HYPERBOLIC PARABOLOID.

EXERCISES

Identify, discuss, and sketch the following surfaces:

1. $9x^2 + y^2 - 9z = 0.$
2. $x^2 - y^2 = 16z.$
3. $4x^2 + 9y^2 - 36z = 0.$
4. $25y^2 + z^2 = 25x.$
5. $x^2 - z^2 = 16y.$
6. $25y^2 + z^2 - x = 0.$
7. $x^2 + y^2 + z^2 = 1.$
8. $-x^2 + y^2 + z^2 = 1.$
9. $y^2 - x^2 - z^2 + 1 = 0.$
10. $4x^2 + 9y^2 + z^2 + 36 = 0.$
11. $4x^2 + 9y^2 + z^2 - 36 = 0.$

XIV.5 IMPROPER QUADRICS—THE CYLINDER—THE CONE

The five types of quadrics so far discussed, the ellipsoid, the hyperboloid of one sheet, the hyperboloid of two sheets, the elliptic paraboloid, and the hyperbolic paraboloid, are called **proper quadrics.**

The cylinder and cone are often called **improper quadrics.** A tangent plane to one of these is tangent along a whole straight line.

We define a cylinder as the surface generated by a straight line that remains parallel to a fixed straight line while one of its points describes a plane curve.

Any section of a cylinder by a plane parallel to its base has the same shape and size as the base curve. Thus, if the generating line is parallel, let us say, to the z-axis, every section, $z = c$, is the same. That is to say, the value of z has no bearing on the equation. (This implies that z does not enter into the equation.)

Conversely, if z is missing from the equation, it can be shown that the surface is a cylinder with all generators parallel to the z-axis.

EXAMPLE 1. The equation

$$4x^2 + 9y^2 = 36,$$

represents a cylinder (Fig. 113) whose base curve, in the xy-plane, is the ellipse having the same equation.

Suppose that a central quadric, when reduced to standard form, has an equation:

$$(1) \qquad ax^2 + by^2 + cz^2 = 0.$$

The surface is a cone, since every plane through the origin intersects the surface in two straight lines, real or imaginary, and any plane parallel to a co-ordinate plane intersects the surface in a conic of the same shape, but whose size is proportional to the distance from the origin.

If a, b, and c all have the same sign, the cone is imaginary, since it has no real points other than the origin. (Some people call it a point-ellipsoid.)

If the algebraic signs are not all the same, the cone is real, with the origin as vertex.

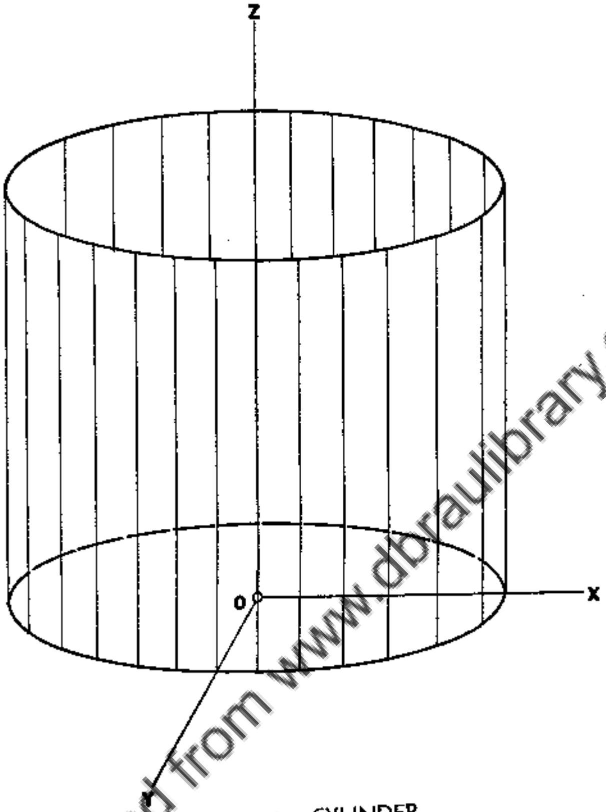

FIG. 113.—CYLINDER.

EXAMPLE 2. Discuss and sketch the surface,

$$4x^2 + y^2 - (z - 4)^2 = 0.$$

If we were to reduce this to standard form, by setting $z - 4 = z'$, the equation would be in the form (1). Hence, the surface is a cone (Fig. 114) with vertex at the new origin, which is (0,0,4) in the original system.

The traces in the xz- and yz-planes are straight lines. The trace in the xy-plane is an ellipse, $4x^2 + y^2 = 16$.

The surface consists of two sheets which extend indefinitely in both directions along the z-axis. Any section made by the plane $z = k (k \neq 4)$ is an ellipse whose center is on the z-axis.

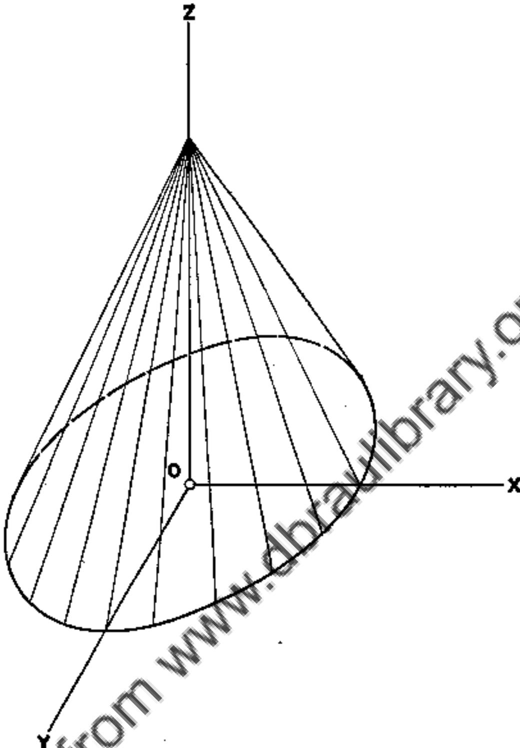

FIG. 114.—CONE.

XIV.6 THE GENERAL QUADRIC

If the equation of a quadric surface is not in one of the standard forms, it can be reduced to standard form by means of transformations similar to those studied in two-dimensional geometry. We shall not discuss them in detail.

EXERCISES

Identify, discuss, and sketch:

1. $16x^2 + 25y^2 = 400.$

2. $y^2 - 4x = 0.$

3. $4x^2 + 9z^2 = 36.$

4. $x^2 - 9y^2 = 36.$

5. $9x^2 + y^2 - z^2 = 0.$

6. $x^2 - y^2 + 16z^2 = 0.$

7. $x^2 + y^2 - (z - 5)^2 = 0.$

8. $4x^2 - (y - 4)^2 + z^2 = 0.$

9. $x^2 - z^2 = 16y.$

10. $25x^2 + 25y^2 + z^2 - 25 = 0.$

11. $4x^2 - 4y + z^2 = 0.$

12. $y^2 - z^2 = 9x.$

13. $x^2 + z^2 = 4y.$

14. $x^2 + y^2 - 9z^2 = 36.$

15. $x^2 - y^2 - z^2 - 1 = 0.$

Sphere

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Ellipsoid

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1$$

Paraboloid

(1) Elliptic

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = cz + d$$

(2) Hyperbolic

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = cz + d$$

Hyperboloid

(1) One sheet

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} - \frac{(z - z_0)^2}{c^2} = 1$$

(2) Two sheets

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} - \frac{(z - z_0)^2}{c^2} = 1$$

n	n^2	$\sqrt{n}$	$1/n$	n	n^2	$\sqrt{n}$	$1/n$
1	1	1	1	51	2601	7.141	.0196
2	4	1.414	.5	52	2704	7.211	.0192
3	9	1.732	.3333	53	2809	7.280	.0189
4	16	2	.25	54	2916	7.348	.0185
5	25	2.236	.2	55	3025	7.416	.0182
6	36	2.449	.1667	56	3136	7.483	.0179
7	49	2.646	.1429	57	3249	7.550	.0175
8	64	2.828	.125	58	3364	7.616	.0172
9	81	3	.1111	59	3481	7.681	.0169
10	100	3.162	.1	60	3600	7.746	.0167
11	121	3.317	.0909	61	3721	7.810	.0164
12	144	3.464	.0833	62	3844	7.874	.0161
13	169	3.606	.0769	63	3969	7.937	.0159
14	196	3.742	.0714	64	4096	8	.0156
15	225	3.873	.0667	65	4225	8.062	.0154
16	256	4	.0625	66	4356	8.124	.0152
17	289	4.123	.0588	67	4489	8.185	.0149
18	324	4.243	.0556	68	4624	8.246	.0147
19	361	4.359	.0526	69	4761	8.307	.0145
20	400	4.472	.05	70	4900	8.367	.0143
21	441	4.583	.0476	71	5041	8.426	.0141
22	484	4.690	.0455	72	5184	8.485	.0139
23	529	4.796	.0435	73	4329	8.544	.0137
24	576	4.899	.0417	74	5476	8.602	.0135
25	625	5	.04	75	5625	8.660	.0133
26	676	5.099	.0385	76	5776	8.718	.0132
27	729	5.196	.0370	77	5929	8.775	.0130
28	784	5.292	.0357	78	6084	8.832	.0128
29	841	5.385	.0345	79	6241	8.888	.0127
30	900	5.477	.0333	80	6400	8.944	.0125
31	961	5.568	.0323	81	6561	9	.0123
32	1024	5.657	.0313	82	6724	9.055	.0122
33	1089	5.745	.0303	83	6889	9.110	.0120
34	1156	5.831	.0294	84	7056	9.165	.0119
35	1225	5.916	.0286	85	7225	9.220	.0118
36	1296	6	.0278	86	7396	9.274	.0116
37	1369	6.083	.0270	87	7569	9.327	.0115
38	1444	6.164	.0263	88	7744	9.381	.0114
39	1521	6.245	.0256	89	7921	9.434	.0112
40	1600	6.325	.025	90	8100	9.487	.0111
41	1681	6.403	.0244	91	8281	9.539	.0110
42	1764	6.481	.0238	92	8464	9.592	.0109
43	1849	6.557	.0233	93	8649	9.647	.0108
44	1936	6.633	.0227	94	8836	9.695	.0106
45	2025	6.708	.0222	95	9025	9.747	.0105
46	2116	6.782	.0217	96	9216	9.798	.0104
47	2209	6.856	.0213	97	9409	9.849	.0103
48	2304	6.928	.0208	98	9604	9.899	.0102
49	2401	7	.0204	99	9801	9.950	.0101
50	2500	7.071	.02	100	10000	10	.01

ANSWERS

II.3, PAGE 10

6. Origin at A: $A(0,0)$, $B(4\sqrt{2},0)$, $C(4\sqrt{2}, 4\sqrt{2})$, $D(0,4\sqrt{2})$.
at B: $A(-4\sqrt{2},0)$, $B(0,0)$, $C(0,4\sqrt{2})$, $D(-4\sqrt{2}, 4\sqrt{2})$.
at C: $A(-4\sqrt{2}, -4\sqrt{2})$, $B(0,-4\sqrt{2})$, $C(0,0)$, $D(-4\sqrt{2}, 0)$.
at D: $A(0,-4\sqrt{2})$, $B(4\sqrt{2},-4\sqrt{2})$, $C(4\sqrt{2},0)$, $D(0,0)$.

II.4, PAGE 12

1. $(2\sqrt{2}, 45°)$, $\left(\sqrt{29}, \text{arc tan } \frac{2}{5}\right)$, $\left(\sqrt{17}, \text{arc tan } -\frac{1}{4}\right)$, $\left(\sqrt{13}, \text{arc tan } \frac{2}{3}\right)$, $(6\sqrt{2}, 135°)$, $(2,0)$, $\left(\frac{\sqrt{13}}{2}, \text{arc tan } -\frac{2}{3}\right)$, $(\sqrt{2}, 90°)$.

3.(a) $\left(6, \frac{5\pi}{4}\right)$; (b) $(-3\sqrt{2}, -3\sqrt{2})$.

II.5, PAGES 14–15

1. $\sqrt{3}$. 3. $\sqrt{5 - 2\sqrt{3}}$. 5. 7. 7. 10.
11.(a) scalene; (b) isosceles; (c) equilateral.

II.6, PAGE 18

2.(a) $P(0,-1)$; (b) $P\left(2, \frac{4}{7}\right)$; (c) $P(14,9)$. 3. $\left(4, \frac{4}{3}\right)$, $\left(7, \frac{8}{3}\right)$.

5.(a) $(7, 0)$; (b) $(4,1)$. 7. Medians intersect in the point $\left(\frac{-5}{3}, \frac{2}{3}\right)$.

9.(a) $\left(6, \frac{19}{4}\right)$; (b) $\left(\frac{36}{5}, \frac{26}{5}\right)$. 10. $\left(\frac{22}{3}, \frac{17}{3}\right)$. 11. $\left(-\frac{1}{3}, \frac{10}{3}\right)$. 12. $\left(-\frac{2}{3}, 3\right)$.

II.7.8, PAGE 23

2.(a) $\frac{3\sqrt{3} - 2}{2}$; (b) $1\frac{3}{8}$.

4.(a) collinear; (b) noncollinear; (c) collinear. 5. $\frac{1}{4}$. 8. $\frac{28}{85}\sqrt{85}$. 10. 4 and 6.

II.9, PAGE 25

1.(a) $m = 1$, $\alpha = 45°$; (b) $m = 1$, $\alpha = 45°$;
(c) $m = -1$, $\alpha = 135°$; (d) $m = \frac{3}{4}$, $\alpha = \text{arc tan } \frac{3}{4}$;
(e) $m = \frac{9}{8}$, $\alpha = \text{arc tan } \frac{9}{8}$; (f) $m = 0$, $\alpha = 0$;
(g) m is infinite, $\alpha = 90°$.
3. Parallel, as slopes are equal.
5. The line segment connecting the mid-points of two sides of a triangle equals one-half of the remaining side.
6.(a) noncollinear, (b) noncollinear, (c) noncollinear.

III.2, PAGE 28

1. $\rho\cos\left(\theta-\frac{\pi}{4}\right)=3$. **3.** $\rho\cos\left(\theta-\frac{2\pi}{3}\right)=1$. **5.** $\rho\cos\left(\theta-\frac{\pi}{2}\right)=6$.

7. $\rho\cos\theta=2$. **9.** $\rho\cos\left(\theta-\frac{\pi}{4}\right)=0$. **11.** No. **13.** Yes.

III.3, PAGE 31

1.(a) $\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}-4=0$; **(b)** $-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}-2=0$; **(c)** $x-5=0$;

(d) $y-1=0$; **(e)** $\frac{\sqrt{3}}{2}x+\frac{1}{2}y-6=0$; **(f)** $\frac{1}{2}x+\frac{\sqrt{3}}{2}y-2=0$;

(g) $\frac{12}{13}x+\frac{5}{13}y-3=0$; **(h)** $\frac{4}{5}x+\frac{3}{5}y-2=0$.

5.(a) $\rho\cos\left(\theta-\frac{\pi}{4}\right)=4$; **(b)** $\rho\cos(\theta-135°)=2$; **(c)** $\rho\cos\theta=5$;

(d) $\rho\cos\left(\theta-\frac{\pi}{2}\right)=1$ or $\rho\sin\theta=1$; **(e)** $\rho\cos(\theta-30°)=6$;

(f) $\rho\cos(\theta-60°)=2$; **(g)** $\rho\cos\left[\theta-\arctan\left(\frac{5}{12}\right)\right]=3$;

(h) $\rho\cos\left[\theta-\arctan\left(\frac{3}{4}\right)\right]=2$.

6.(a) $\rho\cos\left[\theta-\arctan\left(-\frac{4}{3}\right)\right]=3$; **(d)** $\rho\cos\left(\theta-\frac{\pi}{4}\right)=3\sqrt{2}$;

(f) $\theta=\frac{\pi}{4}$.

III.4, PAGES 33–34

1.(a) $-\frac{31}{5}$; **(b)** $-0.9\sqrt{2}$; **(c)** 0; **(d)** $-1.6\sqrt{5}$; **(e)** 0.

3. $x-5y=0$, $5x+y-12=0$. **5.** Result is Formula (6) of §II.7.
7. The bisectors of the interior angles of a triangle intersect in a point.
8. $A(-6,-3)$, $B\left(\frac{30}{7},\frac{9}{7}\right)$, $C(2,3)$. **9.** $x-2=0$, $C\left(2,\frac{1}{3}\right)$.

III.5, PAGES 36–37

1. $7x+y-19=0$. **3.** $2x+3y-4=0$. **5.** $x+1=0$.
7. $\frac{4x}{-5}+\frac{3y}{5}-5=0$. **9.** $\frac{4x}{\sqrt{17}}-\frac{y}{\sqrt{17}}-\frac{3}{\sqrt{17}}=0$.

11. $-\frac{2x}{\sqrt{13}}-\frac{3y}{\sqrt{13}}-\frac{5}{\sqrt{13}}=0$.
13. $6x-5y-20=0$, $2x+3y-16=0$, $10x+y+4=0$.
14.(b) collinear. **15.(b)** E is the point (4,8). **16.** (5,12).

17.(a) Point of intersection $\left(2,\frac{10}{3}\right)$;

(b) Point of intersection $\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$.

III.6, PAGE 38

1. -2. 3. $-\frac{5}{2}$. 5. -1. 7. 0. 9. $\left(x_1, \frac{-Ax_1 - C}{B}\right)$, $\left(x_2, \frac{-Ax_2 - C}{B}\right)$.

III.8, PAGE 41

1. $2x - y - 2 = 0$. 3. $x + y + 1 = 0$. 5. $2x + y - 11 = 0$.
7. $2x - 3y = 0$. 9. $\frac{-2}{5}$. 11.(a) $(a + b, c)$.

III.9, PAGE 43

1.(a) $y = -4x + 5$; (b) $y = \frac{1}{3}x + \frac{4}{3}$; (c) $y = -\frac{5}{7}x + \frac{12}{7}$;
(d) $y = -\frac{A}{B}x - \frac{C}{B}$. 3.(a) $y = 3x + 6$; (b) $y = x - 2$; (c) $y = 5x$;
(d) $y = -x + 3$; (e) $y = -2x - 1$.

III.10, PAGES 45–46

1. arc tan $\left(-\frac{14}{5}\right)$.

3. $\angle A$ = arc tan $\left(\frac{3}{5}\right)$, $\angle B$ = arc tan 3, $\angle C$ = arc tan $\left(\frac{9}{2}\right)$.

5. arc tan $\left(\frac{37}{16}\right)$. 7. arc tan $\left(\frac{1}{2}\right)$. 9. arc tan 3. 10. arc tan $\left(-\frac{8}{5}\right)$.

11. $\frac{2\pi}{3}$. 13. $y = 7x - 15$. 14. $2x + y - 16 = 0$.

III.11, PAGES 47–48

1.(a) parallel; (b) perpendicular; (c) neither; (d) neither; (e) perpendicular; (f) perpendicular.
2.(d) $ax + by - (ax_1 + by_1) = 0$, $bx - ay - (bx_1 - ay_1) = 0$.
3.(a) AB: $4x - 3y - 5 = 0$, BC: $x - 7y + 5 = 0$,
CD: $4x - 3y - 30 = 0$, DA: $x - 7y - 20 = 0$;
(b) $5\sqrt{2}$, (c) $\frac{5}{\sqrt{2}}$; (d) 25, (e) 45°; (f) $7x + y - 15 = 0$.
4. $x - 3y + 27 = 0$, $3x + y - 9 = 0$.
5.(a) Point of intersection $\left(\frac{1}{2}, \frac{9}{2}\right)$;
(b) Equations of perpendicular bisectors:

$$(x_1 - x_2)x + (y_1 - y_2)y + \frac{x_2^2 - x_1^2 + y_2^2 - y_1^2}{2} = 0$$

$$(x_2 - x_3)x + (y_2 - y_3)y + \frac{x_3^2 - x_2^2 + y_3^2 - y_2^2}{2} = 0$$

$$(x_3 - x_1)x + (y_3 - y_1)y + \frac{x_1^2 - x_3^2 + y_1^2 - y_3^2}{2} = 0.$$

6. arc tan $\frac{56}{33}$; bisector: $7x + 56y - 228 = 0$.

IV.1, PAGES 53–54

1.(a) $(x-1)^2+(y-2)^2=36$; (b) $(x+5)^2+y^2=4$;

(c) $x^2+(y+6)^2=3$; (d) $\left(x+\frac{5}{3}\right)^2+\left(y+\frac{1}{2}\right)^2=1$;

(e) $(x+h)^2+(y-k)^2=1$; (f) $\left(x-\frac{4}{5}\right)^2+y^2=16$.

3.(a) $\rho=2$; (b) $\rho=12\cos\left(\theta-\frac{\pi}{3}\right)$; (c) $\rho=8\cos\theta$; (d) $\rho=10\sin\theta$;

(e) $\rho=-10\sin\theta$; (f) $\rho=-8\cos\theta$; (g) $\rho=6\cos\left(\theta+\frac{\pi}{3}\right)$.

4.(a) $C\left(\frac{5}{2},0\right)$, $r=\frac{5}{2}$, on circle; (c) $C\left(4,\frac{\pi}{2}\right)$, $r=4$, not on circle;

(e) $C\left(5,-\frac{\pi}{3}\right)$, $r=5$, not on circle. 5.(a) $(x-6)^2+y^2=36$;

(b) $(x-2)^2+(y+2\sqrt{3})^2=16$; (c) $(x-\sqrt{3})^2+(y-1)^2=4$.

7. $(x-3)^2+y^2=16$.

IV.2, PAGES 55–56

1.(a) $(2,-1)$, $r=\sqrt{6}$; (b) $\left(\frac{1}{3},0\right)$, $r=\sqrt{-\frac{35}{3}}$ (imaginary circle);

(c) $\left(-\frac{3}{4},1\right)$, $r=\frac{1}{4}\sqrt{130}$; (d) $\left(0,-\frac{2}{3}\right)$, $r=0$ (point circle);

(e) $\left(-\frac{a}{2},-\frac{b}{2}\right)$, $r=\frac{1}{2}\sqrt{a^2+b^2-4c}$.

2.(a) $x^2+y^2-4x=0$, $(2,0)$, $r=2$;
(d) $x^2+y^2-8x-8\sqrt{3}y=0$, $(4,4\sqrt{3})$, $r=8$.

3.(a) $\rho=5\cos\left[\theta-\arctan\frac{4}{(3)}\right]$; (b) $\rho=\sqrt{\frac{10}{2}}\cos\left[\theta-\arctan\left(-\frac{1}{3}\right)\right]$.

5. $x^2+y^2-10x-16y+64=0$. 7. Inside.

9.(a) $4x-3y-16=0$, $4x+3y-80=0$.

IV.3, PAGE 57

3. $x^2+y^2+4x-2y-15=0$. 5. $x^2+y^2-4x-17y+45=0$.

IV.4, PAGE 59

1. $(3,4)$, $(-4,-3)$, real and distinct. 3. Both $(4,0)$, real and coincident.

5. $(0,6+2\sqrt{-3})$, $(0,6-2\sqrt{-3})$, imaginary.

6. $(1,1)$, $(2,-1)$, real and distinct.

7. Imaginary, $\left(\frac{46\pm3\sqrt{-6}}{7},\ \frac{25\sqrt{3}\pm6\sqrt{-2}}{7}\right)$.

9. $m=-\frac{1}{2}$. 11. $x-y+1=0$, $x+7y-7=0$.

Review. PAGE 61

1. Collinear. 2. $\rho\cos\left(\theta-\frac{\pi}{4}\right)=8$.

3. $y-4=x+3$ or $y-13=x-6$;

$x-y+7=0$, $\frac{x-y+7}{-\sqrt{2}}=0$, $\rho\cos\left(\theta-\frac{3\pi}{4}\right)=\frac{7}{2}\sqrt{2}$.

5. $\left(4, \frac{10}{3}\right)$. 7.(a) (15,13).

8. $\left|\frac{(y_3 - y_2)\, x_1 - (x_3 - x_2)\, y_1 + x_3\, y_2 - x_2\, y_3}{\pm\sqrt{(y_3 - y_2)^2 + (x_3 - x_2)^2}}\right|$.

9.(a) $\rho = 10 \cos \theta$, $x^2 + y^2 - 10x = 0$;

(b) $\rho = 8 \cos\left(\theta - \frac{\pi}{6}\right)$, $x^2 + y^2 - 4\sqrt{3}\, x - 4y = 0$.

11.(a) $x^2 + y^2 = 25$; (b) $(-4, 3)$. 12. $(x - 12)^2 + (y - 9)^2 = 100$.

13. $2x - 3y + 13 = 0$, $(-2,+3)$, $18x - y - 65 = 0$, $\left(\frac{18}{5}, -\frac{1}{5}\right)$.

15. $(x - 2)^2 + (y - 6)^2 = 5$.

V.1,2, PAGE 64

1. $x^2 - 2xy + y^2 - 14x - 6y + 39 = 0$. 3. $3x^2 + 4y^2 + 6x - 105 = 0$.
5. $25x^2 + 16y^2 + 200x - 106y - 401 = 0$.
7. $y^2 = 4x$. 9. $9x^2 - 16y^2 - 576 = 0$. 11(b) A point. (Circle with 0 radius.)

V.3, PAGE 66

3.(a) $\rho = \frac{10}{1 + \cos\theta}$; (b) $\rho = \frac{6}{1 - \cos\theta}$; (c) $\rho = \frac{45}{1 + 3\cos\theta}$;

(d) $\rho = \frac{8}{3 + 2\cos\theta}$; (e) $\rho = \frac{24}{1 - 2\cos\theta}$; (f) $\rho = \frac{3}{4 + \cos\theta}$.

5. $\rho = \frac{2}{1 + \cos\theta}$.

7.(a) $y^2 + 4x - 4 = 0$; (b) $3x^2 - y^2 - 32x + 64 = 0$;
(c) $3x^2 - y^2 + 24x + 36 = 0$; (d) $3x^2 + 4y^2 - 20x - 100 = 0$.

V.4, PAGES 69–70

1. $y^2 + 4ax = 0$. 2. $x^2 - 4ay = 0$. 3. $x^2 + 4ay = 0$.
5. $F(-2,0)$, $(-2,4)$, $(-2,-4)$, $x - 2 = 0$.
7. $F\left(0, \frac{3}{2}\right)$, $\left(3, \frac{3}{2}\right)$, $\left(-3, \frac{3}{2}\right)$, $2y + 3 = 0$.
9. $F(0, -2)$, $(4, -2)$, $(-4, -2)$, $y - 2 = 0$.
15.(a) (4,4). 16. $(2, \sqrt{6})$, $(2, -\sqrt{6})$. 17. (0,0), $(4, -4\sqrt{2})$.

V.5, PAGE 72

5.(a) $e = \frac{\sqrt{a^2 - b^2}}{a}$; (b) $e = \frac{\sqrt{a^2 + b^2}}{a}$.

6.(a) $\frac{x^2}{9} + \frac{y^2}{25} = 1$; $e = \frac{4}{5}$; $F_1(0,4)$, $F_2(0, -4)$; directrices: $4y - 25 = 0$;

$4y + 25 = 0$; L. R. $\frac{18}{5}$.

(c) $\frac{-x^2}{3} + \frac{y^2}{9} = 1$; $e = \frac{2}{\sqrt{3}}$; foci: $(0, \pm 2\sqrt{3})$; directrices: $y = \pm \frac{3\sqrt{3}}{2}$;

L. R. = 2.

(e) $\frac{-x^2}{9} + \frac{y^2}{9} = 1$; $e = \sqrt{2}$; foci: $(0, \pm 3\sqrt{2})$; directrices: $y = \pm \frac{3}{\sqrt{2}}$;

L. R. = 6.

V.6, PAGES 75–76

2.(a) $\frac{x^2}{36}+\frac{y^2}{27}=1$; (c) $\frac{x^2}{84}+\frac{y^2}{100}=1$; (d) $\frac{x^2}{100}-\frac{y^2}{525}=1$.

6.(a) $\left(1,\pm\frac{3}{2}\right)$; (b) $\left(\pm\sqrt{2},\pm\frac{3}{2}\sqrt{2}\right)$.

VI.1, PAGES 80–81

1. $(-3,0)$, $(-4,-5)$, $(-8,3)$, $(-5,6)$, $(x-4, y-2)$.

3.(a) $3x'+4y'-8=0$; (b) $y'=10x'-4$; (c) $ax'+by'+(c-a)=0$.

4.(a) Circle $x'^2+y'^2=4$; (b) Circle $x'^2+y'^2=9$;
(c) Parabola $y'^2=4x'$; (f) Ellipse $\frac{x'^2}{8}+\frac{y'^2}{24}=1$.

5.(a) Translate to $(1,-2)$, $y'^2=6x'$; (b) Translate to $\left(\frac{33}{16},\frac{3}{2}\right)$, $y'^2=-4x'$;
(c) Translate to $(3,5)$, $x'^2=-10y'$;
(d) Translate to $\left(\frac{-g}{a},\frac{g^2-ac}{2af}\right)$, $x'^2=\frac{2f}{a}y'$.

6.(a) Translate to $(-1,0)$, $\frac{x'^2}{9}+\frac{y'^2}{4}=1$.
(c) Translate to $(1,-1)$, $\frac{x'^2}{4}+\frac{y'^2}{-12}=1$.

7. $(y-1)^2=4(x-4)$; L. R. $=4$; directrix: $x-3=0$.

8. $\frac{(x-2)^2}{25}+\frac{(y+1)^2}{16}=1$; L. R. $=\frac{32}{5}$; foci: $(-1,-1)$, $(5,-1)$;
directrices: $3x-31=0$, $3x-19=0$.

9. $\frac{(x-2)^2}{9}-\frac{y^2}{7}=1$; L. R. $=\frac{14}{3}$; directrices: $4x=17$, $4x=-1$;
center $(2,0)$.

11. $\frac{x^2}{7}+\frac{(y-5)^2}{16}=1$; L. R. $=\frac{7}{2}$; foci $(0,2)$, $(0,8)$; directrix: $3y+1=0$.

13. $\frac{(y-1)^2}{1}-\frac{(x+2)^2}{2}=1$; L. R. $=4$; foci: $(-2,1-\sqrt{3})$, $(-2,1+\sqrt{3})$;
directrices: $y=1-\frac{1}{3}\sqrt{3}$, $y=1+\frac{1}{3}\sqrt{3}$.

15. $(y-k)^2=-4a(x-h)$; L. R. $=4a$; foci: $(h-a,k)$;
directrix: $x=h+a$.

17. $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$; L. R. $=\frac{2b^2}{a}$; directrices: $x=h+\frac{a}{e}$, $x=h-\frac{a}{e}$;
foci: $(h-\sqrt{a^2-b^2},k)$, $(h+\sqrt{a^2-b^2},k)$.

VI.2.3.4, PAGE 86

3. Center $(0,0)$, proper conic.

5. Center $(14,5)$, $x'^2-4x'y'+5y'+2=0$, proper.

6. Center $(14,5)$, conic degenerates into two imaginary lines,
$5y'-(2+i)x'=0$, $5y'+(2+i)x'=0$.

7. Center $(16,18)$, $4x'^2-7x'y'+3y'+11=0$, proper.

9. (b) $x+y-1=0$, $x+y+2=0$;
(c) $x+2y-3=0$, $x+2y-3=0$.

11. Center $\left(2,-\frac{1}{2}\right)$, ellipse $x'^2+2y'^2=\frac{33}{2}$.

13. Center (3,1), ellipse $3x'^2 + 4y'^2 = 12$.
15. Center (5,−2), imaginary circle $x'^2 + y'^2 = -10$.

VI.5, PAGE 90

2. $y = 3x$, $y = -3x$. 3. $y = \frac{x}{2}$, $y = -\frac{x}{2}$.

4. No real asymptotes. 6. $x - y = 0$, $5x - y = 0$.

10. Center $\left(2, -\frac{1}{2}\right)$, hyperbola, $x - y\sqrt{2} - 2 + \frac{1}{2}\sqrt{2} = 0$

and $x + y\sqrt{2} - 2 - \frac{1}{2}\sqrt{2} = 0$.

VI.6, PAGE 93

4. $17x'^2 + 4y'^2 - 68 = 0$, ellipse. 5. $9x'^2 - 4y'^2 - 36 = 0$, hyperbola.
7. $11x'^2 + 121y'^2 - 864 = 0$, ellipse. 9. $5y'^2 - 5x'^2 - 7 = 0$, hyperbola.
11. $y''^2 = -\frac{1}{\sqrt{13}}x''$, parabola. 13. $y''^2 = 2\sqrt{2}x''$, parabola.

Review. PAGE 94

2.(a) $x + 2y - 9 = 0$, (b) $2x - y - 3 = 0$, (c) $x - 2y + 15 = 0$.
Vertices of triangle (3,3), (7,11), (−3,6).

5. (3,3), 8. $\left(-\frac{5}{3}, 1\right)$. $34x''^2 + 9y''^2 = 34$.

VII.1, PAGES 98–99

1. $3x + 4y = 25$, $4x - 3y = 25$. 3. $4x - 3y = -1$, $4x + 5y = -9$.
5. $x + 2y - 7 = 0$, $x - 2y + 5 = 0$. 6. $4x + 3y = \pm 20$.
7. $2x - y = 0$, $2x - y = 20$. 9. $x + 4y = \pm 4\sqrt{10}$.
15. $4x - 3y = \pm 50$. 16. $y + 3x - 27 = 0$.

VII.2, PAGES 101–102

2. $5x + 8y - 21 = 0$, $8x - 5y + 2 = 0$.
3. $x - 9y + 3 = 0$, $9x + y + 27 = 0$.
5. $x + y - 5 = 0$, $x - y + 1 = 0$. 7. $3x - 5y + 23 = 0$, $5x + 3y + 27 = 0$.
9. $xx_1 + yy_1 - r^2 = 0$, $y_1x - x_1y = 0$.
11. $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$, $a^2y_1x + b^2x_1y - (a^2 + b^2)x_1y_1 = 0$.
15. Tangent: $16x + 15y = 100$, Normal: $75x - 80y = 108$.

VIII.1.2, PAGE 104

1. Circle $r = 3$, center at pole. 3. Polar axis.

5. Straight line through $\left(2, \frac{\pi}{6}\right)$. 7. Straight line through $\left(1, \frac{\pi}{2}\right)$.

9. Circle $r = \frac{1}{2}$, center $\left(\frac{1}{2}, \frac{\pi}{2}\right)$.

11.(a) $\rho = \frac{3}{1 + \cos\theta}$; (b) $\rho = \frac{15}{4 + 3\cos\theta}$; (c) $\rho = \frac{12}{2 + 3\cos\theta}$.

13. $\rho = \frac{4}{1 - \cos\theta}$ (parabola). 15. $\rho = \frac{5}{1 + \frac{5}{4}\cos\theta}$ (hyperbola).

VIII.5.6.7, PAGE 110

16. $\rho^2 \sin 2\theta = a^2$. 17. $\rho = -4 \sin \theta$. 18. $\rho = a(1 - \cos \theta)$.
19.(1) $(x^2 + y^2)^3 = 4(x^2 - y^2)^2$. (5) $4(x^2 + y^2)^3 - 4(x^2 + y^2)^2 + y^2 = 0$.
(7) $x^2 + y^2 - y = 0$. (11) $(x^2 + y^2)^2 = 9(x^2 - y^2)$.
(14) $y^2 = \dfrac{x(x-1)^2}{2-x}$.

21. $\left(a, \dfrac{\pi}{2}\right)$, $\left(a, -\dfrac{\pi}{2}\right)$. 23. $\left(2, \pm \text{arc tan}\dfrac{1}{3}\right)$, $\left(2, \pm \text{arc tan} -\dfrac{1}{3}\right)$.

X.1,2, PAGE 128

7.(1) $y = x^3$. (2) $y^2 - x + 2y + 1 = 0$. (3) $x^2 - y^3 = 0$.
(4) $2xy - y - 3 = 0$. (5) $9x^2 - 6xy + y^2 + 14x - 38y + 64 = 0$.
(6) $x^3 - 3x^2y + 3xy^2 - y^3 - 4x - 4y = 0$ or $(x - y)^3 = 4(x + y)$.
9. $x^3 + y^3 - 3axy = 0$.

X.4, PAGE 132

5.(3) $\left(\dfrac{x}{5}\right)^{2/3} + \left(\dfrac{y}{3}\right)^{2/3} = 1$. (4) $(4 - y^2)^3 = 4x^2$.

XII.2, PAGES 154–155

1. 6. 3. 10. 5. $d = \sqrt{x_1^2 + y_1^2 + z_1^2}$. 6.(a) (2,2,0); (b) $\sqrt{13}$ and 7.
9.(a) Collinear; (b) Collinear; (c) Collinear.
10. $\left(\dfrac{5}{2}, 4, 6\right)$. 12. $\sqrt{41}$, $\sqrt{77}$, $\sqrt{56}$.

XII.3, PAGES 156–157

1. $\dfrac{1}{\sqrt{59}}$, $\dfrac{3}{\sqrt{59}}$, $\dfrac{7}{\sqrt{59}}$. 3. $\dfrac{2}{\sqrt{29}}$, $\dfrac{-4}{\sqrt{29}}$, $\dfrac{3}{\sqrt{29}}$.

5. (1, 0, 0; 0, 1, 0; 0, 0, 1). 7. No. 9. $\left(3, -1, \dfrac{9}{2}\right)$.

11.(a) Parallel to xy-plane, $\perp$ to z-axis;
(b) Parallel to xz-plane, $\perp$ to y-axis;
(c) Parallel to x-axis, $\perp$ to yz-plane.

XII.4.5, PAGE 159

1. Arc cos $\dfrac{1}{\sqrt{3}}$. 3. Arc cos $\dfrac{11}{21}$. 5. $\dfrac{\pi}{2}$. 7(a) No; (b) No.

XII.6, PAGE 162

1. $45\frac{1}{6}$. 3. Zero (gives points coplanar).

5. No. 6. $x + 6 = 0$. 7. In the plane $4x - 3y + 10z - 17 = 0$.

XIII.1.2, PAGES 166–167

1. $A = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$, $B = \begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$, $C = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$.
3.(a) Noncoplanar; (b) Coplanar.
7.(a) yz-plane, xz-plane, xy-plane;
(b) $\parallel$ to yz-plane, $\perp$ to yz-plane, $\parallel$ to xy-plane;
(c) All $\perp$ to xy-plane.

XIII.3, PAGE 169

1. $x + y + \sqrt{2}z - 10 = 0.$ 2. $x - \sqrt{2}y - z + 8 = 0.$ 3. $x - 3 = 0.$
4. $x - \sqrt{3}y + 10 = 0.$ 5. $x - \sqrt{2}y - z + 6 = 0.$
7. $\parallel$ to z-axis, $\perp$ to xy-plane, $3x + 2y - 6 = 0.$
9. The yz-plane, $x = 0.$
11. $x - 4y + 8z - 9 = 0$, meeting each of the axes in a point that is not the origin.

XIII.4.5, PAGE 172

1. $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, p = 1.$ 3. $0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, p = \frac{1}{2\sqrt{5}}.$
5. $\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, p = \frac{1}{\sqrt{6}}.$
7. $\frac{a}{-\sqrt{a^2+b^2+c^2}}, \frac{b}{-\sqrt{a^2+b^2+c^2}}, \frac{c}{-\sqrt{a^2+b^2+c^2}}, p = \frac{d}{+\sqrt{a^2+b^2+c^2}}$
(for positive values of both a and d).
$\frac{a}{+\sqrt{a^2+b^2+c^2}}, \frac{b}{+\sqrt{a^2+b^2+c^2}}, \frac{c}{+\sqrt{a^2+b^2+c^2}}, p = \frac{d}{-\sqrt{a^2+b^2+c^2}}$
(for positive values of a and negative values of d).
8. $d = -2\sqrt{6}$ (on same side). 9. $d = \frac{-4}{\sqrt{14}}$ (on same side).
11. $d = -\frac{150}{\sqrt{769}}$ (on same side). 13. $d = \frac{60}{\sqrt{178}}$ (on same side).
15. $d = \frac{10}{\sqrt{14}}.$

XIII.6.7, PAGE 174

1. $x + 3y + 5z - 19 = 0.$ 3. $4x - 2z - 8 = 0.$ 5. $6x - y + 6 = 0.$
7. $x + y + \sqrt{2}z - 3(1 + \sqrt{2}) = 0.$ 9. $x + 2y + 5z - 30 = 0.$
10. $3x + 4z - 25 = 0.$ 12. $\frac{x}{2} + \frac{y}{-3} + \frac{z}{6} = 1,\ 3x - 2y + z - 6 = 0.$
13. $\frac{x}{1} + \frac{y}{\frac{2}{3}} + \frac{z}{-\frac{4}{5}} = 1,\ 4x + 6y - 5z - 4 = 0.$
14. $\frac{x}{p/q} + \frac{y}{r/s} + \frac{z}{q/p} = 1,$
$$q^2rx + pqsy + p^2yz = pqr.$$
15. $\frac{x}{5} + \frac{y}{-2} + \frac{z}{\frac{40}{23}} = 1,\ 8x - 20y + 23z - 40 = 0.$

XIII.8.9, PAGES 177–178

1. x-axis: $y = 0, z = 0$; y-axis: $z = 0, x = 0$; z-axis: $x = 0, y = 0.$
3. 13, 16, 7. 4. 13, −15, 6. 5. 7, −21, −5.
9. Direction cosines of each line are $\frac{6}{101}, \frac{-8}{101}, \frac{1}{101}.$

XIII.10, PAGE 180

1. $-2, -1, 3$ (direction numbers); $\frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ (direction cosines).

3. $\frac{x-1}{3} = \frac{y-2}{-3} = \frac{z-3}{-3}$. 5. $\frac{x-3}{1} = \frac{y-2}{5} = \frac{z}{-8}$.

7. $\frac{x}{-3} = \frac{y-1}{5} = \frac{z+2}{1}$. 9. $2x + 2y + z - 5 = 0$.

10. $\frac{x}{3} = \frac{y-1}{4} = \frac{z-2}{5}$.

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Review Exercises
Chapters XII and XIII.

1. $\frac{1}{3}\sqrt{161}$.

3. $3x - 3y - 2z + 18 = 0$, $\frac{x}{-6} + \frac{y}{6} + \frac{z}{9} = 1$, $\frac{3x - 3y - 2z + 18}{-\sqrt{22}} = 0$.

5. $-\frac{54}{13}$, 39. 7. $x - y + 3z - 16 = 0$. 9. $(2,4,6)$.

10.(a) $3y + z - 18 = 0$; (b) $2x - y = 0$; (c) $6x + z - 18 = 0$.

XIV.1, PAGE 184

1. $(x-1)^2 + (y-4)^2 + (z+3)^2 = 36$, $x^2 + y^2 + z^2 - 2x - 8y + 6z - 10 = 0$.
3. $x^2 + y^2 + z^2 - 59 = 0$. 5. $x^2 + y^2 + z^2 - 10z = 0$.
7. $x^2 + y^2 + z^2 - 2x - 4y - 10z + 14 = 0$. 9. Center $(0,0,0)$, radius $= 5$.
11. Center $(4,2,0)$, radius $= 5$. 13. Given point is inside the sphere.
15. Each sphere is outside the other.

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